Display stacks

Mathematical Tables and other Aids to Computation

QA 47 .M43

UNIVERSITY OF MICHIGAN
MAY 20 3
MATHEMATICS LIBRARY

A Quarterly Journal

Edited by

E. W. CANNON
C. C. CRAIG
A. ERDÉLYI
F. J. MURRAY
J. TODD
D. H. LEHMER, *Chairman*

VII • Number 42 • April, 1953 • p. 73–134

Published by

THE NATIONAL RESEARCH COUNCIL
Washington, D. C.

NATIONAL RESEARCH COUNCIL

DIVISION OF MATHEMATICS

EDITORIAL COMMITTEE

E. W. CANNON (E.W.C.), National Bureau of Standards, Washington, D. C. Automatic Computing Machinery [ACM].

C. C. CRAIG (C.C.C.), University of Michigan, Ann Arbor, Mich. Mathematical Statistics [K].

A. ERDÉLYI (A.E.), California Institute of Technology, Pasadena, Calif. Higher Mathematical Functions [L].

F. J. MURRAY (F.J.M.), Columbia University, New York, N. Y. Other Aids to Computation [OAC].

J. TODD (J.T.), National Bureau of Standards, Washington, D. C. Numerical Methods.

D. H. LEHMER (D.H.L.), *Chairman*, National Bureau of Standards, Los Angeles 24, California.

SUBSCRIPTION RATES

1943–1945: (Nos. 1–12) $12.00 for all 12 issues (not available for separate sale except as noted below*)

1946–1949: $4.00 per year

1950–1953: $5.00 per year

Single issues are available for sale as follows:

* 1944 (No. 7, "A Guide to Tables of Bessel Functions," by H. Bateman and R. C. Archibald, 104 pp.) $2.00

1946–1949 (Nos. 13–28) $1.25 for single issue

1950–1953 (Nos. 29–44) $1.50 for single issue

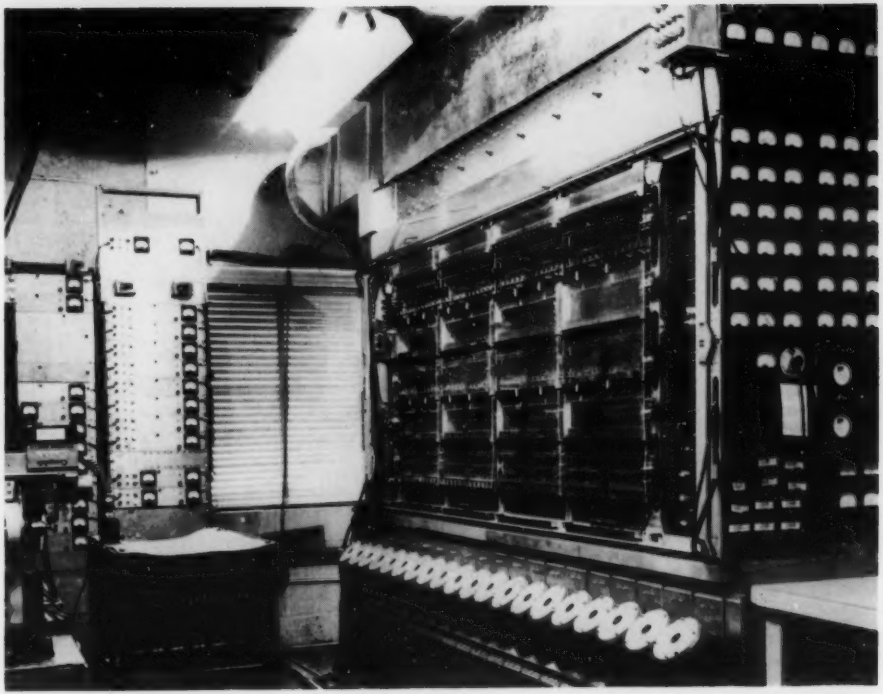
All payments are to be made to National Academy of Sciences, 2101 Constitution Avenue, Washington, D. C.

Agents for Great Britain and Ireland (subscription 42s, 6d for 1953) Scientific Computing Service, Ltd., 23 Bedford Square, London W.C.1.

Published quarterly in January, April, July and October by the National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation*, and all Books for review, should be addressed to D. H. Lehmer, 16556 Chattanooga Place, Pacific Palisades, Calif.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.



INSTITUTE FOR ADVANCED STUDY COMPUTER

A Technique for Real Time Simulation of a Rigid Body Problem

1. Introduction. With the advent of large-scale high-speed digital computers, there arises the question of their possible use in the solution of problems in "real time," i.e., in conjunction with instruments receiving and responding to stimuli from the external environment. The criteria for satisfactory operation in such real-time service are different from those generally encountered. First, the computer must calculate fast enough to respond to the fastest changing stimuli; speed in solving the associated equations is in fact the primary requirement. Again, because it is impossible accurately to predict stimuli variations, either the response must be delayed or a "reasonable" amount of inaccuracy must be tolerated or both. Third, because of the limited range of number representation in the digital computer (or, for that matter, in any computer), all parameters must remain bounded no matter when or where they occur. Finally, the computations must be stable.

This paper is devoted to a discussion of the manipulation of the dynamical equations of a rotating rigid body with fixed center of gravity.[1] The techniques here adapted will be seen to sacrifice accuracy to speed of computation, but in such a way as to hold inaccuracies reasonably low; at the same time, a singularity generally encountered will be circumvented. Stability will not be discussed.

2. Manipulation of Eulerian angles. The equations of motion of a rotating rigid body are usually simplified by using a set of axes attached to the rigid body. The simplification results from the fact that this choice of axes causes the moments and products of inertia to become constants. In addition, the products of inertia can be made to vanish by properly orienting these axes with respect to the rigid body.

The orientation of the rotating axes OX, OY, and OZ relative to a set of fixed axes OX_0, OY_0, and OZ_0 may be defined by the Eulerian angles ψ, θ, ϕ. For convenience assume OZ_0 downward, OX_0 North, and OY_0 East. ψ is the angle measured about OZ_0 from the X_0Z_0 plane to the (vertical) plane containing OZ_0 and OX; it is positive when OX is toward the East. θ is the angle between OX and the X_0Y_0 plane; it is positive when OX is above the X_0Y_0 plane. It is measured about an axis in the X_0Y_0 plane. Call this axis OY'. ϕ is the angle measured about the OX axis from OY' to OY; it is positive for clockwise rotation as viewed from the origin along the positive OX axis.

If the angular velocity of the moving axes has components p, q, and r about the moving axes OX, OY, and OZ, respectively, then the following three differential equations show the relations among the angles and the components of angular velocity.

$$\dot{\theta} = q \cos \phi - r \sin \phi,$$

$$\dot{\phi} = \frac{1}{\cos \theta} (p \cos \theta + q \sin \theta \sin \phi + r \sin \theta \cos \phi),$$

$$\dot{\psi} = \frac{1}{\cos \theta} (q \sin \phi + r \cos \phi).$$

The presence of $\cos\theta$ in the denominator makes two of these equations indeterminate when θ is $\pm 90°$. This makes the equations difficult to use especially where point by point numerical computations are being performed. One common method is to limit the variable θ say to the range $\pm 85°$. This is avoiding the problem rather than solving it.

This indeterminism is a product of the mathematical approach to a physical problem. A different choice of axes for example would present the same difficulty at a different orientation of the rotating body. Therefore the way out of the difficulty required a new mathematical point of view.

3. Derivation of direction cosine equations. When the equations of motion contain the Eulerian angles they appear in combinations equal to one or more of the direction cosines. That is, the Eulerian angles can be eliminated from the mechanical equations of motion by introducing the direction cosines as parameters. The kinematic differential equations for the direction cosines are quite simple and have no indeterminate points. This is the principal gain.

The new differential equations can be derived by the following method. The unit vector along the X_0 axis is related to the unit vectors along the moving X, Y, Z axes by

$$i_0 = il_1 + jm_1 + kn_1.$$

Differentiate this with respect to time

$$0 = \frac{di}{dt} l_1 + i\dot{l}_1 + \frac{dj}{dt} m_1 + j\dot{m}_1 + \frac{dk}{dt} n_1 + k\dot{n}_1.$$

If the moving axes are rotating with an angular velocity $ip + jq + kr$

$$\frac{di}{dt} = jr - kq, \qquad \frac{dj}{dt} = kp - ir, \qquad \frac{dk}{dt} = iq - jp.$$

Substituting these values for the derivatives we obtain

$$i(\dot{l}_1 - rm_1 + qn_1) + j(\dot{m}_1 - pn_1 + rl_1) + k(\dot{n}_1 - ql_1 + pm_1) = 0,$$

which requires that

$$\begin{aligned} \dot{l}_1 &= rm_1 - qn_1, \\ \dot{m}_1 &= pn_1 - rl_1, \\ \dot{n}_1 &= ql_1 - pm_1. \end{aligned}$$

A similar treatment of j_0 and k_0 yields similar equations with subscripts 2 and 3, respectively.

4. Identities. Since there were three independent differential equations involving the Eulerian angles only three of these nine new differential equations can be expected to be independent of the 21 identities that obtain for direction cosines. These 21 identities include six expressions for normality

$$l_1^2 + m_1^2 + n_1^2 = 1, \text{ etc.}, \tag{1}$$

six for orthogonality

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0, \text{ etc.} \tag{2}$$

and nine of the form

$$l_1 = m_3n_3 - m_3n_2, \text{ etc.} \tag{3}$$

To choose three independent equations, choose equations all of which do not have the same subscripts nor do all involve the time derivative of the same letter. For example

$$l_1 = rm_1 - qn_1,$$
$$l_2 = rm_2 - qn_2,$$
$$\dot{m}_2 = pn_2 - rl_2$$

is a set of independent equations which, with six of the identities, can be used to obtain all nine direction cosines. If desired, more than three differential equations and fewer than six identities can be used. For some applications, such as analogue computers, the differential equations are more suitable than the identities. Even for digital computers the differential equations may be preferred to identities of the form of equations (1) and (2).

5. Computation Procedure. Before discussing the plan of computation, it will be well to discuss a fundamental change that must often be made. Since there is no unity in many modern digital computers, this number cannot be held in the machine. Indeed, if a direction cosine were computed correctly to be unity, it would appear in such a machine as zero. To remove this difficulty, pseudo direction cosines can be introduced, which are M times the corresponding true direction cosines. For example, with $M = 1/2$, the identities are of the form

$$l_3^2 + m_3^2 + n_3^2 = \tfrac{1}{4} \qquad \text{(6 of these)},$$
$$l_3l_2 + m_3m_2 + n_3n_2 = 0 \qquad \text{(6 of these)},$$
$$l_1 = 2(m_2n_3 - m_3n_2) \qquad \text{(9 of these)}.$$

The first step in the proposed computation procedure is the use of three differential equations to compute l_3, m_3, n_3. Since both curtailment and round-off errors will gradually cause these parameters to fail to obey the normality equations, a second step is required to correct this failure.

Let the computed values be l_3', m_3', n_3'. Define ϵ by the equation:

$$l_3'^2 + m_3'^2 + n_3'^2 = M^2(1 + \epsilon).$$

A normalized set of direction cosines is therefore

$$l_3^2 = \frac{l_3'^2}{1 + \epsilon}, \qquad m_3^2 = \frac{m_3'^2}{1 + \epsilon}, \qquad n_3^2 = \frac{n_3'^2}{1 + \epsilon}.$$

By using two terms in the binomial expansion of $(1 + \epsilon)^{-\frac{1}{2}}$,

$$l_3 = l_3'\left(1 - \frac{\epsilon}{2}\right) = 2l_3'\left(\frac{1}{2} - \frac{\epsilon}{4}\right), \text{ similarly } m_3, n_3,$$

thus avoiding the square root computation.

l_2', m_2', n_2' may now be computed from the appropriate three differential equations. Consider the two vectors,

$$r_3 = il_3 + jm_3 + kn_3,$$
$$r_2' = il_2' + jm_2' + kn_2'.$$

The vector r_3 has magnitude M. The vector r_2' should, but in general will not, have magnitude M and be perpendicular to r_3. Let r_2' be turned through as small an angle as possible to make it perpendicular to r_3. To do this it is necessary only to look for r_2 in the plane of r_2' and r_3, i.e.,

$$r_2 = (1 + a)r_2' + br_3.$$

The scalars a and b must at the same time make the magnitude of r_2 equal to M.

Define ϵ_1 and ϵ_2 by the equations

$$l_2'^2 + m_2'^2 + n_2'^2 = M^2(1 + \epsilon_1),$$
$$l_2'l_3 + m_2'm_3 + n_2'n_3 = M^2\epsilon_2.$$

If the errors ϵ_1 and ϵ_2 are small, the corrections a and b will be small; then it will be sufficiently exact to neglect second degree terms in a, b, ϵ_1, ϵ_2. Requiring r_2 to have magnitude M leads to

$$a = -\frac{\epsilon_1}{2},$$

while the fact that r_2 is normal to r_3 yields

$$b = -\epsilon_2.$$

Thus, neglecting second order terms, the corrected values may now be written

$$l_2 = l_2'\left(1 - \frac{\epsilon_1}{2}\right) - l_3\epsilon_2,$$

$$m_2 = m_2'\left(1 - \frac{\epsilon_1}{2}\right) - m_3\epsilon_2,$$

$$n_2 = n_2'\left(1 - \frac{\epsilon_1}{2}\right) - n_3\epsilon_2.$$

The correction involving ϵ_1 adjusts the magnitude while the correction involving ϵ_2 corrects the direction of r_2. The results obtained show that to the first order either correction may be applied first or they may both be applied together.

There is a second method of orthogonalizing, which is preferable because it is quicker. In this method the computed values of two of l_2, m_2, n_2 are assumed correct and the third is adjusted to make r_2 perpendicular to r_3. If n_3 is greater than either l_3 or m_3, for example, only l_2 and m_2 are computed by integrating their differential equations; n_2 is then determined from the orthogonality condition

$$n_2 = \frac{l_2l_3 + m_2m_3}{-n_3}.$$

Note that if n_2' had been computed from the differential equation, then

$$l_2l_3 + m_2m_3 + n_2'n_3 = \epsilon_3M^2,$$
$$n_2 = \frac{n_2'n_3 - \epsilon_3M^2}{n_3} = n_2' - \frac{\epsilon_3M^2}{n_3}.$$

This amounts to correcting n_2' by the term

$$-\frac{\epsilon_2 M^2}{n_3}.$$

Obviously if l_2' had been corrected instead of n_2' the correction would be

$$-\frac{\epsilon_2 M^2}{l_3}.$$

To make the correction small, the largest of l_3, m_3, n_3 is made to appear in the denominator. This also prevents division by zero, which might occur if n_2 were corrected under all circumstances.

After the direction of r_2 has been established perpendicular to r_3, its magnitude must be adjusted. The procedure is to alter all components proportionately as for r_3 above. Clearly this does not destroy orthogonality. l_1, m_1, n_1 can now be computed from equation (3)

$$l_1 = \frac{m_2 n_3 - m_3 n_2}{M}, \text{ etc.}$$

6. Conclusions. The equations of motion of a rotating rigid body can be formulated directly in terms of the direction cosines. The direction cosines can be determined as described above in a manner which is thought to be ideally suited to digital computation. No trigonometric functions of the Eulerian angles appear, obviating the necessity of referring to a function table or to a series expansion to obtain the solutions of the equations of motion. In addition, all quantities are finite for all orientations of the rigid body.

This work was done in connection with a study performed by the University of Pennsylvania, Moore School of Electrical Engineering, under Contract N6Onr 24913 sponsored by the Office of Naval Research, Special Devices Center, Port Washington, New York.

H. J. Gray, Jr.
M. Rubinoff
H. Sohon

University of Pennsylvania
Moore School of Electrical Engineering

[1] Leigh Page, *Introduction to Theoretical Physics*, Second Edition—Fourteenth Printing, D. Van Nostrand Company, Inc., New York, N. Y., Introduction, Chap. I, II; 1951.

A Solution of Simultaneous Linear Equations and Matrix Inversion with High Speed Computing Devices

In solving several systems of simultaneous linear equations, in which the coefficients of the unknowns are the same in all the systems but in

which the constant terms vary from system to system, it is advantageous to invert the matrix formed by the coefficients of the unknowns. This inverted matrix, common to all the systems, is then multiplied by the vector formed by each set of constant terms, and a set of solutions is thus obtained for each of the above systems,[1] or simultaneous solutions of the systems obtained at the time of inversion. The inversion of a matrix, especially if it is of a high order, is a laborious and time-consuming process. From time to time new methods of inversion have been developed and modified to suit the computation equipment available until it seems that a saturation point in simplicity of application, accuracy of results, and speed of computation has been reached, and further improvements will have to come from the development of the computing machines themselves or from their application. For information on these topics the reader is referred to items 2–12 of the bibliography.

The following is a modification of an algebraic treatment of the matrix by the direct method which will reduce the labor and time necessary for inversion. This method has time and labor saving characteristics applicable to computations with low and high memory computing devices, making the inversion of high order matrices less forbidding.

In addition to the customary definitions of inversion, pivotal row, leading element, etc., which are here taken for granted, we will define prepivotal and transpivotal rows as the rows respectively preceding and following the pivotal row. Consider also the original matrix as extended to the right by means of the identity matrix and a check column (other augmentations of the original matrix may be included to suit the needed results[6] and a check row may be added for ease in locating the source of spurious results).

In a simplified method[11] of inversion the elements of this extended matrix are operated upon by certain reduction operations and an intermediate matrix is obtained, the elements of which, on a second application of the reduction formulas, yield another intermediate matrix, and so on until, after the nth reduction, the inverted matrix is obtained. The reduction operations usually consist of a quotient operation applied to the elements of the pivotal row and a binomial operation applied to the elements of the other rows. This reduction cycle has a definite advantage in its simplicity, but it involves computations with the elements of the transpivotal rows in all the previous reductions, in preparation of values for the time when these transpivotal rows become pivotal or prepivotal. Also, with automatic computing machines, all the elements of the identity matrix are involved in the calculations even at the time when some of them do not contribute to the final result.

The method proposed in this article computes partial reductions (on some of the rows only) on pivotal, prepivotal, and the first transpivotal rows, until the memory capacity of the computing equipment is reached, at which point a complete reduction (on all the rows of the matrix) is computed, repeating this cycle, with the last obtained matrix, until final reduction (inversion) is reached; all this while, at a given time only the relevant contributing part of the identity matrix is used. It is to be noted that the saving in the identity matrix is independent of the transpivotal row economy and may be used with other inversion methods as well.

Three types of computing operations are used in each matrix reduction.

1. Divisional or quotient operation, used on the elements of the pivotal row.
2. Binomial operation, used on the elements of the prepivotal rows.
3. Polynomial operation, used on the elements of the first transpivotal row for partial matrix reductions, and on all the transpivotal rows for complete matrix reductions.

Let the elements of the original extended matrix be x_{ij} (where i and j respectively represent the row and column of the particular element, i may assume values from 1 to n, and j values from 1 to $2n + 1$). Let the elements of the matrix developed after the first, second . . . and nth reductions be $a_{ij}, b_{ij} \cdots n_{ij}$, respectively. As usual the ith row is pivotal for the ith reduction. The formulas for the first five reductions are:

(1)
$$a_{1j} = x_{1j}/x_{11} \qquad (j = 2(1)n + 1)$$
$$a_{2j} = x_{2j} - x_{21}a_{1j}$$

(2)
$$b_{1j} = a_{1j} - a_{12}b_{2j}$$
$$b_{2j} = a_{2j}/a_{22} \qquad (j = 3(1)n + 2)$$
$$b_{3j} = x_{3j} - x_{32}b_{2j} - x_{31}b_{1j}$$

(3)
$$c_{1j} = b_{1j} - b_{13}c_{3j}$$
$$c_{2j} = b_{2j} - b_{23}c_{3j} \qquad (j = 4(1)n + 3)$$
$$c_{3j} = b_{3j}/b_{33}$$
$$c_{4j} = x_{4j} - x_{43}c_{3j} - x_{42}c_{2j} - x_{41}c_{1j}$$

(4)
$$d_{1j} = c_{1j} - c_{14}d_{4j}$$
$$d_{2j} = c_{2j} - c_{24}d_{4j}$$
$$d_{3j} = c_{3j} - c_{34}d_{4j} \qquad (j = 5(1)n + 4)$$
$$d_{4j} = c_{4j}/c_{44}$$
$$d_{5j} = x_{5j} - x_{54}d_{4j} - x_{53}d_{3j} - x_{52}d_{2j} - x_{51}d_{1j}$$

(5)
$$e_{1j} = d_{1j} - d_{15}e_{5j}$$
$$e_{2j} = d_{2j} - d_{25}e_{5j}$$
$$e_{3j} = d_{3j} - d_{35}e_{5j} \qquad (j = 6(1)n + 5)$$
$$e_{4j} = d_{4j} - d_{45}e_{5j}$$
$$e_{5j} = d_{5j}/d_{55}$$
$$e_{6j} = x_{6j} - x_{65}e_{5j} - x_{64}e_{4j} - x_{63}e_{3j} - x_{62}e_{2j} - x_{61}e_{1j}$$

The check column should be included in the above values of j.

Thus, on the first reduction, the pivotal row is the first row, the elements of which are divided and the binomial operation is applied to the elements of the second row. On the second reduction, the second, first, and third rows are pivotal, prepivotal, and the first transpivotal, respectively. In the third reduction, the third row is pivotal, the first and the second rows are prepivotal, and the fourth row is the first transpivotal, etc.

With an unlimited, or sufficiently large, memory machine the process is continued until the nth reduction, when the nth row is pivotal, the rows less than n are prepivotal and there are no transpivotal rows. In applying the above formulas to the elements of the original (extended) and intermediate matrices, only columns of the identity matrix up to and including the unit element of the corresponding pivotal row need be included.

With limited memory machines the process is continued until the memory capacity is reached, at which point the quotient operation is applied to the elements of the pivotal row, the binomial operation is used on the elements of the prepivotal rows and the polynomial operation is computed for the elements of all the transpivotal rows. Thus, if in the above equations the fourth reduction had been complete, the general equation for the transpivotal row elements would have been:

$$d_{ij} = x_{ij} - x_{i4}d_{4j} - x_{i3}d_{3j} - x_{i2}d_{2j} - x_{i1}d_{1j} \qquad (i = 5(1)n)$$

With a minimum memory machine all the reductions are computed as usual with a quotient operation on the elements of the pivotal row and a binomial operation on the elements of all the other rows. Unlimited, limited, and minimum memory machines are left undefined except in their use to classify the three broad applications of the method as described above.

The polynomial expression for the transpivotal rows, which is easily obtained algebraically from previous binomial reductions by a rearrangement of terms and by a substitution of pivotal and prepivotal results of the same column, permits partial reductions of the matrix due to the fact that it depends only on the pivotal and prepivotal results of the same column and on the leading elements of the original extended matrix for the corresponding row.

It is to be noticed that in the above formulas we must distinguish between two types of leading elements: (a) derived leading elements for pivotal and prepivotal rows and taken from the previous matrix reduction, and (b) original leading elements for transpivotal rows and taken from the original extended matrix (or from an intermediate matrix obtained in a previous complete reduction). For a given transpivotal row, there are as many original leading elements as the number of the reduction. Varying degrees of continuity within the cycle itself may be obtained according to the storing capacity, programming facilities, available means for selection of properly coded rows and columns, and other characteristics of the available equipment.

Between the two extremes of no memory and unlimited memory there is an optimum program design which is approached by a judicious choice of continuity of operation using the above minimum memory requirements to meet the available memory capacity.

We have successfully and economically applied the above method to the inversion of 20 by 20 matrices using a full capacity 602-A.IBM calculating punch while using words of 12 digits throughout in order to take care of creeping decimal errors from the right and overflowing on the left. Three partial and one complete reductions per cycle were obtained.

With reduced numbers, when possible, greater savings may be achieved, but they are not advisable when handling high order matrices. Even when

large integral values may be avoided by preliminary or intermediate treatment of the matrix, the creeping decimal error is ever present and may be met only with otherwise unnecessary decimals or with iterative methods.[8]

The time consumed by this method and the time by the method of all complete reductions were in the ratio of 1 to 3 when using the same equipment in both cases and with the same number handicap, by the same operators.

GABRIEL G. BEJARANO
BRUCE R. ROSENBLATT

California Research Corporation
Richmond, California

[1] A. C. AITKEN, *Determinants and Matrices*. Edinburgh, 5th ed., 1948.
[2] P. S. DWYER, *Linear Computations*. New York, 1951.
[3] H. HOTELLING, "Some new methods in matrix calculation," *Annals Math. Stat.*, v. 14, 1943, p. 1–34.
[4] P. D. CROUT, "A short method for evaluating determinants and solving systems of linear equations with real or complex coefficients," AIEE, *Trans.*, v. 60, 1941, p. 1235–1241.
[5] G. W. PETRIE III, "Matrix inversion and solution of simultaneous linear algebraic equations with the IBM type 604 Electronic Calculating Punch," IBM Computation Seminar, August 1951, *Proc.*, p. 105–111.
[6] J. CHANCELLOR, J. W. SHELDON, & G. L. TATUM, "The solution of simultaneous linear equations using the IBM Card-Programmed Electronic Calculator," IBM Industrial Computation Seminar, September 1950, *Proc.*, p. 57–61.
[7] W. D. BELL, "Punched card techniques for the solution of simultaneous equations and other matrix operations," IBM Scientific Computation Forum, 1948, *Proc.*, p. 28–31.
[8] J. VON NEUMANN & H. H. GOLDSTINE, "Numerical inverting of matrices of high order," Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 1021–1099.
[9] A. OPLER, "Monte Carlo matrix calculation with punched card machines," *MTAC*, v. 5, 1951, p. 115–120.
[10] J. SHERMAN, "Computations of inverse matrices by means of IBM machines," IBM *Technical Newsletter*, no. 3, 1951, p. 22–25.
[11] J. LOWE, "Solution of simultaneous linear algebraic equations using the IBM type 604 Electronic Calculating Punch," IBM Computation Seminar, Dec. 1949, *Proc.*, p. 54–56.
[12] F. M. VERZUH, "The solution of simultaneous linear equations with the aid of the 602 Calculating Punch," *MTAC*, v. 3, p. 453–462.

Error Bounds on Approximate Solutions to Systems of Linear Algebraic Equations

1. The Abmatrix. Consider the real matrix $A = (a_{ij})$, $i = 1(1)m$, $j = 1(1)n$. Define $\alpha(A)$, read "the abmatrix of A," to be

$$\alpha(A) = (|a_{ij}|).$$

Thus, $\alpha(A)$ is the *matrix* having each element equal to the absolute value of the corresponding element of A. In particular, for a scalar s,

$$\alpha(s) = |s|.$$

If $A = (a_{ij})$ and $B = (b_{ij})$, $i = 1(1)m$, $j = 1(1)n$, define

$$\alpha(A) \leqslant \alpha(B)$$

to mean

$$|a_{ij}| \leqslant |b_{ij}| \qquad \text{for all } i, j.$$

Thus, if $\alpha(A) \leqslant \alpha(B)$ and $\alpha(B) \leqslant \alpha(C)$, then $\alpha(A) \leqslant \alpha(C)$.

From these definitions, it follows that if the operations $A + B$ and AB are permissible, then

$$\alpha(A + B) \leqslant \alpha(A) + \alpha(B), \tag{1}$$

and

$$\alpha(AB) \leqslant \alpha(A)\alpha(B). \tag{2}$$

Extensions to and combinations of (1) and (2) are indicated. In particular, if A is a square matrix and p is a positive integer,

$$\alpha(A^p) \leqslant [\alpha(A)]^p.$$

Subsequently, $[\alpha(A)]^p$ will be written $\alpha^p(A)$.

This simple notion of the abmatrix is readily applicable to examining the convergence of a class of iterative methods for solving systems of linear algebraic equations and to determining error bounds for approximate solutions of these linear systems; however, in this note, only the latter will be examined. For their application, the use of an approximate inverse is required.

2. The Error in the Approximate Solution. Let the linear system under consideration be

$$A\xi = \eta,$$

where A is non-singular and of order n. Let G be an approximate inverse of A, the approximate solution being

$$x = G\eta.$$

Then,

$$\xi = \sum_{h=0}^{p} (I - GA)^h x + (I - GA)^{p+1}\xi, \tag{3}$$

which is simply proven by induction.

Denote now the error in x by $\epsilon = \xi - x$, and let $D = I - GA$. Thus, from (3),

$$\epsilon = \sum_{h=1}^{p} D^h x + D^{p+1}\xi.$$

Restrict G by the condition that $D^p \to 0$ as $p \to \infty$. Then,

$$\epsilon = \sum_{h=1}^{\infty} D^h x, \tag{4}$$

and this expression will be the basis for placing bounds on the error ϵ.

3. Requirements on D and First Uses of the Abmatrix. In Part 2 it was required that $D^p \to 0$ as $p \to \infty$. By means of the abmatrix, easy sufficient conditions for this convergence can be found.

Let $\gamma = (g_i)$, $i = 1(1)n$, be a column vector with $g_i > 0$. It is always possible to find a scalar k, depending on γ, such that

$$\alpha(D)\gamma \leqslant k\gamma. \tag{5}$$

It then follows that

$$0 \leqslant \alpha(D^p)\gamma \leqslant \alpha(D^{p-1})\alpha(D)\gamma \leqslant k\alpha(D^{p-1})\gamma.$$

Thus, by induction,

$$0 \leqslant \alpha(D^p)\gamma \leqslant k^p\gamma.$$

Hence, if $k < 1$, $D^p \to 0$ as $p \to \infty$.

In conclusion, (5) with $k < 1$ is a sufficient condition for the desired convergence. Clearly, the same arguments hold for a row vector γ'.

Suppose then that $k < 1$ for some specified column vector γ. Bounds on $\sum_{h=1}^{\infty} D^h$ may be found by column-wise examination. Consider $\sum_{h=1}^{\infty} D^h e_j$ where e_j is the jth column of I. Then,

$$\alpha\left[\sum_{h=1}^{\infty} D^h e_j\right] \leqslant \sum_{h=1}^{\infty} \alpha(D^h e_j).$$

Since

$$\alpha(D)e_j \leqslant f_j\gamma,$$

where f_j is a scalar depending on γ, it follows that

$$\alpha\left[\sum_{h=1}^{\infty} D^h e_j\right] \leqslant \sum_{h=0}^{\infty} f_j k^h \gamma,$$

and for $k < 1$,

$$\alpha\left[\sum_{h=1}^{\infty} D^h e_j\right] \leqslant \frac{f_j}{1-k}\gamma. \tag{6}$$

Similarly, for some specified row vector with $k < 1$,

$$\alpha\left[\sum_{h=1}^{\infty} e_i' D^h\right] \leqslant \frac{f_i}{1-k}\gamma'. \tag{7}$$

4. Error Bounds on the Approximate Solution. The application of (6) and (7) to stating error bounds on the approximate solution is immediate. It follows from (4) that

$$\alpha(\epsilon) \leqslant \left[I + \alpha\left(\sum_{h=1}^{\infty} D^h\right)\right]\alpha(Dx), \tag{8}$$

and (8) may be coupled with (6) or (7). It is true that (8) requires the matrix multiplication Dx, but considering that the larger matrix multiplication GA is required anyhow, it appears wise to benefit from the usually sharper error bounds given by (8) at the expense of some incremental work. In addition, (3) indicates that Dx is the first order correction to the approximate solution x, and thus Dx can also be used to improve x.

If Dx is not computed, (4) may be used as follows:

$$\alpha(\epsilon) \leqslant \alpha\left[\sum_{h=1}^{\infty} D^h\right]\alpha(x), \tag{9}$$

and (9) may be coupled with (6) or (7).

Finally, x may be taken to be a column of G, the approximate inverse, and thus, error in G may be also investigated.

5. Specific Example of Error Bounds on the Approximate Solution. As yet, no mention has been made of how γ should be chosen once D is given. Generally speaking, the choice would be to minimize k, but this indeed is a serious problem.[1] Without looking for the optimum choice, inspection of D may give some guide. However, a simple choice for γ is $e = \sum_{j=1}^{n} e_j$. For example, with $D = (d_{ij})$, let

$$k = \max_j \sum_i |d_{ij}|,$$

$$f_i = \max_j |d_{ij}|.$$

It is then seen that

$$e'\alpha(D) \leqslant ke',$$

$$e_i'\alpha(D) \leqslant f_i e'.$$

If $k < 1$, (7) gives

$$\alpha\left[\sum_{h=1}^{\infty} e_i' D^h\right] \leqslant \frac{f_i}{1-k} e',$$

whence from (9)

$$\alpha(e_i'\epsilon) \leqslant \frac{f_i}{1-k} e'\alpha(x). \tag{10}$$

Clearly, with the methods presented, many other error bounds such as (10) are possible.

A. DE LA GARZA

Carbide and Carbon Chemicals Company
K-25 Plant, Oak Ridge, Tennessee

[1] The referee makes the following comments:

"If all $|d_{ij}| > 0$, the optimum choice of γ is unique and is the eigenvector u_1 of $\alpha(D)$ whose components are all positive, and k is then the dominant eigenvalue λ_1 of $\alpha(D)$. This follows from a lemma that, since all $g_i > 0$, λ_1 lies strictly between the minimum and maximum of the ratios $e_i'\alpha(D)\gamma/e_i'\gamma$, unless the ratios are all equal (and hence equal to λ_1). The lemma is a slight extension of Theorem I of HAZEL PERFECT, 'On matrices with positive elements,' *Quart. Jn. of Math.*, s. 2, v. 2, 1951, p. 286–290.

"The vector $\alpha^p(D)e$, which is asymptotically a multiple of u_1 as $p \to \infty$, may be a useful approximation to γ for sufficiently large p."

RECENT MATHEMATICAL TABLES

1065[A].—H. S. UHLER, "Many-figure approximations for $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt[3]{4}$, and $\sqrt[3]{9}$ with χ^2 data," *Scripta Math.*, v. 18, 1952, p. 173–176.

Values of the constants mentioned in the title are given to 710, 709, 478, and 478 decimals respectively.

The author reports errata in the values of the cube roots of 2 and 3 as given by BOORMAN.[1] These are correct only to 198D and 54D instead of the 305D and 241D given.

There is given also data on the distribution of the digits of each of the four cube roots. Values of χ^2 and the associated probabilities are given for the first 50 k digits for $k = 1, 2, \cdots$. No unusual distributions are revealed.

D. H. L.

[1] J. M. BOORMAN, "Square-root notes," *The Mathematical Magazine*, v. 1, 1887, p. 208.

1066[D, R].—HEINZ WITTKE, (1) *Geodätische Registertafeln. Schnell-Rechentafel. Fünfstellige Winkelfunktionen. 400^g mit direkter zweistelliger Interpolation und Fehlergrenzen.* Goslar-Harz, Selbstverlag, 1952, 60 p., 16.2 × 24 cm., 9.80 Marks bound. (2) *Geodätische Registertafeln Schnell-Rechentafel. Sechsstellige Winkelfunktionen. 400^g zur Maschinenrechnungen.* Goslar-Harz, Selbstverlag, 1953, 59 p., 16.7 × 25.5 cm., 14.00 Marks bound.

These very neatly prepared geodesy tables, each with many label flap guides, contain tables of sin, tan, cot, cos, for interval hundredth of a grade, differences being printed in red. In each is introductory material with illustrative examples. In (1) are a few useful supplementary tables; and in (2) are sine and cosine tables for [0(0^g.1)100^g; 11D].

R. C. ARCHIBALD

Brown University
Providence, R. I.

1067[F].—H. GUPTA, "A table of values of $N_3(t)$," East Panjab Univ., *Research Bulletin*, no. 20, 1952, p. 13–93.

The table gives for each possible integer $t \leq 20000$ all non-negative integral solutions x, y of

$$x^2 + y^2 = t. \tag{1}$$

There is also given the sum function

$$N_3(t) = \sum_{j=0}^{t} n_3(j)$$

where $n_3(t)$ denotes the number of solutions (x, y) of (1), the solutions (x_0, y_0) and (y_0, x_0) being counted as different when $x_0 \neq y_0$.

This table is mentioned as existing in manuscript in *MTAC*, v. 3, p. 22, and was used to prepare the author's table of $N_3(t)$, the number of solutions of

$$x^2 + y^2 + z^2 \leq t.$$

The author states that

$$N_3(t) = \sum_{0 \leq i \leq t^{\frac{1}{2}}} [(t - i^2)^{\frac{1}{2}}] \tag{2}$$

but this is in contradiction with the definition of $N_3(t)$ and its tabulated values. In fact the terms

$$1 + [t^{\frac{1}{2}}]$$

should be added to the right hand side of (2).

The table can be used to find the exact number of lattice points inside or on a circle of radius r by means of the formula

$$4N_3(r^2) - 4r - 3.$$

Thus the number of lattice points inside and on a circle of radius 140 is

$$4N_3(19600) - 560 - 3 = 61529.$$

The area of the circle is greater than this by the factor 1.000751.

D. H. L.

1068[F].—OVE HEMER, *On the Diophantine Equation* $y^2 - k = x^3$. Uppsala, 1952, 101 p., 23.5 × 17.6 cm.

The booklet is primarily concerned with the solutions of the equation

$$y^2 - k = x^3$$

for positive $k < 100$.

Table 1 (p. 94) purports to give all solutions for 47 of the 59 values of $k < 100$ for which solutions exist (10 of these were cases which were known previously), together with the number N of solutions. For $k = 37$, however, N is given as 2, but besides the two solutions which are listed, the solution (243, 3788) can be read from ROBINSON's tables, which are on deposit in the UMT File [*MTAC*, v. 5, p. 162], and of which the author is apparently unaware. The discussion pertaining to the case $k = 37$ (p. 75–78) is therefore incomplete. In ten of the remaining 12 cases one solution is given and it is stated that $N \leq 2$. In the other 2 cases $k = 63$ and $k = 76$, two and one solutions are given respectively with the remark that "in all probability there are no more solutions."

There is only a one page discussion (p. 88–89) for negative k. A statement is made on p. 88 that $y^2 + 56 = x^3$ has no solutions. However, Robinson's table lists (18, 76) which is indeed a solution. Table 2 (p. 95) lists the 44 values of $k < 0$ for which solutions exist, together with the known solutions and their number. In 22 of the cases all solutions are known, in the remaining cases one or two solutions are given. An extra solution can be read from Robinson's table for $-k = 7, 28, 39, 47, 53, 55, 60$, and 63.

Table 3 gives solutions of $y^2 \pm 27k = x^3$ for $|k| \leq 50$. These were obtained with the help of CASSELS'[1] table of rational solutions [*MTAC*, v. 4, p. 202]. All solutions are supposed to be given, except for 11 values of k. For $k = -11$, however, the number of solutions is given as 8, while a ninth solution (1362, 50265) can be read from ROBINSON'S table. In two other cases, $k = 24$ and 37 which are not complete, Robinson gives an extra solution.

Table 4 gives the discriminants $-D$, the fundamental rings and units ϵ in the cubic field corresponding to 36 square free values of k between 2 and 100. In 10 of these cases ϵ is not "definitely proved to be a fundamental unit."

EMMA LEHMER

16556 Chattanooga Pl.
Pacific Palisades, Calif.

[1] J. W. S. CASSELS, "*The rational solutions of the Diophantine equation* $y^2 = x^3 - D$," *Acta Math.*, v. 82, 1950, p. 243–273.

1069[F].—M. KRAITCHIK, "On the factorization of $2^n \pm 1$," *Scripta Math.*, v. 18, 1952, p. 39–52.

This expository article contains tables giving the complete factorization of $2^n \pm 1$ for those values of n for which such information was known to the author. There are four tables. The first is for $2^n - 1$ and for

$$n = 1(2)99, 105, 107, 111(2)117, 123, 127, 135.$$

The second is for $2^n + 1$ and for

$$n = 1(2)99, 105, 111, 123, 135.$$

The third is for $2^n + 1$ and for

$$n = 4(4)100, 108, 120.$$

The fourth is for $2^n + 1$ and for

$$n = 2(4)130, 138(4)162, 170(4)190, 198, 210, 222, 234, 250, 258, 270, 330.$$

The tables separate the algebraic factors from the primitive ones. The fourth table has column headings for the two factors in

$$2^{4k+2} + 1 = (2^{2k+1} - 2^{k+1} + 1)(2^{2k+1} + 2^{k+1} + 1).$$

These two factors are inadvertently interchanged for $n = 4k + 2 = 94$, 114, and 150.

An additional entry can be given, namely,

$$2^{89} + 1 = 3 \cdot 179 \cdot 6202089 \cdot 1858477404602617.$$

The factor 6202089 was discovered by the SWAC. The large factor was proved prime by A. L. BROWN and the reviewer independently.

Other additional entries may be made from Notes **131**, **138**, **142**, and **146**.

D. H. L.

1070[I].—J. M. HAMMERSLEY, "Lagrangian integration coefficients for distance functions taken over right circular cylinders," *Jn. Math. Phys.*, v. 31, 1952, p. 139–150.

The author considers the problem of evaluating the sixfold integral of the form

$$I = \int t(r) dv_1 dv_2 / r^2$$

where dv_1 and dv_2 are two volume elements, a distance r apart, in a cylinder of diameter a and length ca and $t(r)$ is an arbitrary function of r. The integral is approximated by

$$I = (a/2)^4 \pi^2 (cy_1 + y_2)$$

where

$$y_1 = \int_0^c t(a\xi) g(\xi) \xi^{-2} d\xi,$$

$$y_2 = \int_0^c t(a\xi) h(\xi) \xi^{-2} d\xi + t(ac) k(c).$$

The auxiliary functions g, and k involve the hypergeometric function

$$F(\alpha, 1/2; 2; x) \qquad (\alpha = -1/2, -3/2).$$

The integrals y_1 and y_2 are to be evaluated by the Lagrangian integration formulas

$$y_1 = \sum t(a\xi) G(\xi),$$

$$y_2 = \sum t(a\xi) H(\xi),$$

the sums being taken over equally spaced points ξ.

The author tabulates $G(\xi)$, $-H(\xi)$ and (for checking purposes) $G(\xi) + H(\xi)$ for

$$\xi = 0(.01)2(.05)3(.25)10; 8D$$

with the last place uncertain.

G. BLANCH

NBSINA

1071[K].—T. W. ANDERSON & D. A. DARLING, "Asymptotic theory of certain 'goodness of fit' criteria based on stochastic processes," *Annals Math. Stat.*, v. 23, 1952, p. 193–212.

The table contained in this paper (p. 203) gives the argument z to 5D which satisfies $a_1(z) = \lim_{n\to\infty} \Pr(n\omega^2 \leqq z)$ for $a_1(z) = .01(.01).99, .999$. This is the limiting distribution for large samples of $n\omega^2$, a statistic attributed to VON MISES for testing the hypothesis that n independent, identically distributed random variables have a specified distribution function $F(x)$. ω^2 is defined by $\omega^2 = \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 dF$ where $F_n(x)$ is the sample cumulative distribution function for a sample of size n.

W. J. DIXON

University of Oregon
Eugene, Oregon

1072[K].—Z. W. BIRNBAUM, "Numerical tabulation of the distribution of Kolmogorov's statistic for finite sample size," Amer. Stat. Assn., *Jn.*, v. 47, 1952, p. 425–441.

Let $X_1, X_2, \cdots, X_N$ be a random sample from a population having a continuous cumulative distribution function $F(x)$. Define the empirical distribution function $F_N(x)$ by $F_N(x) = j/N$ where j is the number of observed values of $X_1, \cdots, X_N$ which are less than or equal to x, and let

$$D_N = \text{l.u.b.}_x \, |F(x) - F_N(x)|.$$

Table I (p. 428–430) gives values to 5D of $P(D_N < c/N)$ for $N = 1(1)100$ and $c = 1(1)15$. Table II (p. 431) gives, for $N = 2(1)5(5)30(10)100$, values to 4D of the 95th and 99th percentiles of the distribution of D_N and compares these with the limiting values.

FRANK MASSEY

University of Oregon
Eugene, Oregon

1073[K].—A. H. BOWKER & H. P. GOODE, *Sampling Inspection by Variables.* McGraw-Hill Book Co., Inc., New York, 1952. xi + 216 p., 15.6 × 23.3 cm. Price $5.00.

This book is built around a set of single and double variables sampling inspection plans of the non-central t type. For normal variables the cases of known and unknown standard deviation, and two-sided as well as one-sided specifications, are included. No sequential variables plans are described, for reasons of incomplete theory and elaborate calculation.

For roughly corresponding discussion of single, double (and sequential) attributes sampling inspection plans, see Statistical Research Group, Columbia University, *Sampling Inspection*, New York, McGraw-Hill Book Company, Inc., 1948.

The heart of this book lies in Chapter 11 (pages 125–141) on the mathematics of the plans, and in tables and charts (pages 164–209), together with a number of other tables and charts scattered throughout the text. In addition, there is much material on sampling inspection generally, such as the installation and administration of sampling inspection plans, tightened and reduced plans, disposition of rejected product, non-normal populations, control charts in sampling inspection, comparisons of variables and attributes plans.

A few of the principal tables: Table A (necessary for later tables): for each of fourteen inspection lot sizes and for each of three inspection levels, a sample size letter (42 entries, 14 different). Table B: for single sampling, for each of the fourteen sample size letters (each here corresponding to a single sample size) and for each of fifteen AQL classes, ranging unevenly from .024–.035 to 8.5–11.0, the value of k to 3D in the acceptance criterion $\bar{x} + ks \leqq U$ or $\bar{x} - ks \geqq L$, where U and L are the upper and lower specification limits respectively, and AQL, $\bar{x}$, and s have the usual meanings. (The acceptable quality level, AQL, as here used, is the percentage of defective items in an inspection lot such that the inspection plan will accept 95% of all lots submitted containing that percentage of defects.) Table C: as in Table B, for double sampling, showing k_a, k_r, and k_t, all to 3D, corresponding to k in Table B, but for acceptance on the first sample, rejection on the first sample, acceptance on the second sample. Table D: for each of the fourteen sample size letters and for each of the fifteen AQL classes, the AQL and the LTPD. (The lot tolerance percent defective, LTPD, as here used is the percentage of defective items in an inspection lot such that the sampling plan will reject 90% of all lots submitted containing that percentage of defects.) Table E (the major table of the book): for each of the fourteen sample size letters, the operating characteristic curves of each of from 9 to 16 single and double sampling plans, each pair of plans showing the AQL class, the AOQL class, and the criteria for acceptance (163 curves in all). (The average outgoing quality limit, AOQL, is the maximum percent defective remaining in lots subjected to a plan if all rejected lots are screened, no matter what may be the quality of the lots submitted.)

This work is a sturdy accomplishment by the authors and a credit to all who pioneered in this area. Among the latter are H. ROMIG, H. DODGE, N. L. JOHNSON, B. L. WELCH, W. J. JENNETT, C. C. CRAIG, W. A. WALLIS, K. J. ARNOLD, J. WOLFOWITZ, and the two authors.

H. A. FREEMAN

Massachusetts Institute of Technology
Cambridge, Massachusetts

1074[K].—K. FREUDENBERG, "Die Grenzen für die Anwendbarkeit des Gesetzes der kleinen Zahlen," *Metron*, v. 16, 1951, p. 285–310.

The principal objective of this paper is to determine the smallest sample size required in order to obtain a given degree of approximation to the

binomial distribution by means of the Poisson distribution and by means of the following, so to speak, second order approximating function

$$(1) \qquad B_{\lambda,m} = \frac{e^{-\lambda}\lambda^m}{m!}\left[1 - \frac{(m-\lambda)^2 - m}{2s}\right],$$

where s is sample size, $\lambda = sq$ is the mean number of occurrences of an event of probability q in random samples of size s; λ is also the mean of the approximating Poisson distribution and $m \equiv$ an integer. The mean of $B_{\lambda,m}$ is also independent of s, and its variance is $\lambda(1 - \lambda s^{-1})$. In spite of the fact that the zeroth moment of $B_{\lambda,m}$ is unity independent of s, it is not a distribution, since some of its values can be negative. Since these negative probabilities are, however, exceedingly small, they do not vitiate the practical usefulness of $B_{\lambda,m}$ as an approximation to the Poisson and binomial distributions. In fact, the $B_{\lambda,m}$ require much smaller sample sizes for equivalent approximation to the binomial distribution than does the Poisson distribution. The Poisson and binomial are in this notation denoted as follows:

$$(2) \qquad A_{\lambda,m} = e^{-\lambda}\lambda^m/m!$$

$$(3) \qquad C_{\lambda,m} = \binom{s}{m}\left(\frac{\lambda}{s}\right)^m\left(1 - \frac{\lambda}{s}\right)^{s-m}.$$

The following tables are of principal interest: on p. 307 values of $s_{\rho,A}$ and $s_{\rho,B}$ as functions of $\lambda = \frac{1}{2}(\frac{1}{2})10$, which are respectively lower bounds for s such that the deviations of $A_{\lambda,m}$ and $B_{\lambda,m}$ from $C_{\lambda,m} \leqq \rho = .001$. $s_{\rho,A}$ varies from 141.5 to 640.8, where correspondingly $s_{\rho,B}$ goes from 7.8 to 78.8. On p. 308 $s_{\rho,A}$ and $s_{\rho,B}$ are given as functions of $q = .01(.01).10$, again for $\rho = .001$. Here $s_{\rho,A}$ varies from 406.0 to 4912.2 while $s_{\rho,B}$ goes from 0 to 34.1.

Auxiliary tables of some interest are found on p. 292–294. $A_{\lambda,m}$, $B_{\lambda,m}$, $C_{\lambda,m}$, $(A_{\lambda,m} - C_{\lambda,m})$, and $(B_{\lambda,m} - C_{\lambda,m})$ are given for $s = 50$ and 100 and for $\lambda = \frac{1}{2}, m = 0(1)7, > 7; \lambda = 1, \frac{3}{2}, m = 0(1)10, > 10; \lambda = 2, m = 0(1)11, > 11; \lambda = 3, m = 0(1)12, > 12; \lambda = 4, m = 0(1)13, > 13$.

For the approximation $B_{\lambda,m}$ to be of practical use in sampling, extensive tables would be required; the present table has limited usefulness. The reader's attention may be called to errors in formula (11) p. 289 and the subsequent argument. The right side of (11) should be λ and the subsequent argument can be corrected so as to yield this result. The findings of the paper are not affected by these errors.

S. B. LITTAUER

Columbia University
New York City, New York

1075[K].—C. H. GOULDEN, *Methods of Statistical Analysis*. John Wiley & Sons, New York. 2nd ed., 1952, vii + 467 p., 15.2 × 23.7 cm. $7.50.

The tables in this volume present no novelty, consisting of areas and ordinates of the normal curve, percentage points for the Student-Fisher t, χ^2, and the variance ratio, values of the bias factor c_2 for the standard deviation in samples from normal, and an extract from the FISHER & YATES random numbers.

C. C. C.

1076[K].—A. HALD, *Statistical Tables and Formulas*. London, Chapman & Hall; New York, John Wiley & Sons, 1952. 97 p., 27.8 × 21.9 cm., $2.50.

The actual tables are preceded by 22 pages of pertinent formulas, explanations and illustrations. Tables I and II of ordinates and cumulative frequencies of the normal frequency function has the unusual feature of giving all values to 4S for the standardized argument $u = 0(.01)4.99$. Table III is a 3D probit table slightly abridged from FISHER & YATES.[1] The usual percentage points of the Student-Fisher t distribution are replaced by "fractiles" (Table IV) which are percentage points of the cumulative distribution. These are taken from the Fisher & Yates tables with the omission of 55% fractiles and the addition of 99.9% values as well as the set of values for 50, 80, 100, 200 and 500 degrees of freedom. Table V, fractiles of the χ^2 distribution, is reproduced from the HALD & SINKBAEK table[2] and gives values of χ_0^2 satisfying $P(\chi^2 < \chi_0^2) = p$ to 1D for $100p = .05, .1, .5, 1, 2.5, 5, 10(10)90, 95, 97.5, 99, 99.5, 99.9, 99.95$ and the degrees of freedom $f = 1(1)100$. Table VI gives fractiles of χ^2/f to 4D for $100p = .05, .1, .5, 1, 2.5, 5, 95, 97.5, 99, 99.9, 99.95$ and $f = 1(1)100(5)200(10)300(50)1000(1000)5000, 10000$. Table VII, fractiles of the variance ratio, $\chi_1^2 f_2/\chi_2^2 f_1$, is an extension both of the Fisher & Yates tables and of those of MERRINGTON & THOMPSON.[3] Values are given to 3S for $100p = 50, 70, 90, 99.9, 99.95$ with $f_1 = 1(1)10, 15, 20, 30, 50, 100, 200, 500, \infty$; $f_2 = 1(1)20(2)30(10)60, 80, 100, 200, 500, \infty$ and for $100p = 95, 97.5, 99, 99.5$ with $f_1 = 1(1)20(2)30(5)50, 60, 80, 100, 200, 500, \infty$; $f_2 = 1(1)30(2)50(5)70(10)100, 125, 150, 200, 300, 500, 1000, \infty$. Table VIII for fractiles of the range in samples from a normal universe has been reproduced in part from TIPPETT[4] and PEARSON.[5] Here values are given to 2D for $100p = .05, .1, .5, 2.5, 5, 10(10)90, 95, 97.5, 99, 99.5, 99.9, 99.95$ and the sample size $n = 2(1)20$. Values of the mean, standard deviation and their ratio for the range in samples from normal are given to 3D for the same set of sample sizes.

The next two tables are devoted to functions useful in the estimation of the mean, ξ, and the standard deviation, σ, of a one-sided truncated normal distribution with a known point of truncation according to methods given in HALD's text[6] of which these tables form the companion volume. In the case only a sample of n from the truncated range is used, for arguments y, calculable from the sample, $z = f(y)$ gives an estimate of the standardized point of truncation, ζ; another function, $s = \bar{x}g(z)$, where $\bar{x}$ is the sample mean, estimates σ; and $\bar{x}$ estimates ξ. Table IX gives z to 3D for $y = .55(.001).91$, $g(z)$ with its first differences to 4D for $z = -3(.1)2$, and the variances and covariance of $n^{\frac{1}{2}}\bar{x}/\sigma$ and $n^{\frac{1}{2}}s/\sigma$ to 4S and their coefficient of correlation to 3D also for $z = -3(.1)2$. For a one-sided censored normal distribution the sample consists of $n - a$ known values falling to the right of the known truncation point and a of whose values no more is known than that they fall to the left of the point of truncation. In this case ζ is estimated by $z = f(h, y)$ where $h = a/n$ and y is calculated from the known portion of the sample, σ is estimated from $\bar{x}g(h, z)$, where $\bar{x}$ is the mean of the known portion of the sample and $g(h, z)$ is simply obtained from the tabulated function $\psi'(z)$ and h, and again $\bar{x}$ estimates ξ. In Table X, $f(h, y)$ is given to 3D for $h = .05(.05).5$ and $y = .5(.005).6(.01).8(.05)1.5$; $\psi'(z)$ to 5S with

first and second differences, the same variances and covariances as before to 4S and the same coefficient of correlation to 3D, all for $z = -3(.1)2$.

Table XI gives two-sided 95% and 99% confidence limits for the probability ϑ in the binomial distribution if the sample consists of x occurrences in n trials. Values are given to 3D for x and $n - x = 0(1)20(2)30(5)50, 60, 80, 100, 200, 500, \infty$. Table XII gives 2 arc sin $\sqrt{x}$ to 4D for $x = 0(.001)1$. Tables XIII of log $(n!)$ to 4D for $n = 1(1)1000$, XV of n^2 for $n = 0(1)999$, XVI of $n^{\frac{1}{2}}$ and $(10n)^{\frac{1}{2}}$ for $n = 1(.01)9.99$, XVII of n^{-1} to 5D for $n = 1(.01)9.99$, and XVIII a 4D table of common logarithms are all readily found elsewhere. But Table XIV of log $\binom{n}{x}$ to 4D for $x = 1(1)\left(\frac{n}{2}\right)$ and $n = 2(1)100$ appears to be an extension of previous tables. And the final Table XIX is a new set of 15000 random digits compiled from drawings in the Danish State Interest Lottery.

This is an excellent set of statistical tables containing a good deal of new material.

C. C. C.

[1] R. A. Fisher & F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research*. London & Edinburgh, 1948.

[2] A. Hald & S. A. Sinkbaek, "A table of percentage points of the χ^2-distribution," *Skandinavisk Aktuarietidskrift*, v. 33, 1950, p. 168–175. [*MTAC*, v. 7, p. 25.]

[3] M. Merrington & C. M. Thompson, "Tables of percentage points of the inverted beta (F) distribution," *Biometrika*, v. 33, 1943, p. 73–88. [*MTAC*, v. 1, p. 78.]

[4] L. H. C. Tippett, "On the extreme individuals and the range of samples taken from a normal population," *Biometrika*, v. 17, 1925, p. 364–387.

[5] E. S. Pearson & H. O. Hartley, "The probability integral of the range in samples of n observations from a normal population," *Biometrika*, v. 32, 1942, p. 301–308. [*MTAC*, v. 1, p. 105.]

[6] A. Hald, *Statistical Theory with Engineering Applications*. New York & London, 1952.

1077[K].—T. Kitagawa, *Tables of Poisson Distribution*. Tokyo, Japan, Baifukan, 1951, 156 p., 18.7 × 26.0 cm., $3.50.

The integer-valued random variable X is said to have a Poisson distribution with parameter m if $\Pr(X = x) = e^{-m}m^x/x! = p(x, m)$.

The tables give $p(x, m)$ for $x = 0(1)\cdots$, $m = .001(.001)1$ to 8D (Table I); $m = 1.01(.01)5$ to 8D (Table II); and $m = 5.01(.01)10$ to 7D (Table III).

These tables were prepared directly from their definition. No interpolation methods are suggested.

In the introductory material there is an English translation of a Japanese article by Kitagawa on the subject of double sampling inspection plans which illustrates some of the uses of the above tables.

In Table IV 95% and 99% confidence limits are given to 6D for the population parameter, m, with the sample mean $\bar{x} = .001, .005, .01, .02(.02).1, .2(.2)1(1)10$ and the sample sizes $50(10)100(100)500, 1000, 2000(2000)10000, 20000, 50000, 100000$, when large sample theory is applicable.

Kitagawa prepared his tables before it was possible for him to learn about Molina's tables.[1] In these $p(x, m)$ is given to 6D for $m = .001(.001)-.01(.01).3(.1)15(1)100$ together with $P(c, m) = \sum_{x=c}^{\infty} p(x, m)$ to 6D for the same values of m and $c = 0(1)153$. Kitagawa's volume does not contain a

table of either $P(c, m)$ or $1 - P(c + 1, m)$. In about one hundred comparisons between Molina's and Kitagawa's tables no discrepancies were found.

I. R. SAVAGE

National Bureau of Standards
Washington, D. C.

[1] E. C. MOLINA, *Poisson's Exponential Binomial Limit*. New York, 1942. [*MTAC*, v. 1, p. 18.]

1078[K].—K. KUNISAWA, H. MAKABE & H. MORIMURA, "Tables of confidence bands for the population distribution function. 1.," Union of Japanese Scientists & Engineers, *Reports of Statistical Application Research*, v. 1, 1951, p. 23–44.

Let $(X_1, \cdots, X_n)$ be a sample of independent variables with the same continuous cumulative distribution function $F(x)$, and let $N(z)$ be the number of X_k's which are $\leqq z$. Setting $F_n^*(x) = N(z)/n$, the maximum D_n of $|F_n^*(z) - F(z)|$ is a random variable. Then KOLMOGOROV,[1] who introduced this statistic, showed that $n^{\frac{1}{2}}D_n$ has the asymptotic cumulative probability function $L(z) = 1 - 2\sum_{j=1}^{\infty} (-1)^{j-1}e^{-2j^2z^2}$. This function was tabulated by N. SMIRNOV[2] to 6D, for arguments $z = .28(.01)3$. In the present paper the authors publish a table of $L(z)$ to 7D, for $z = .250(.001)2.299$, tabulating also the first differences.

Z. W. BIRNBAUM

University of Washington
Seattle, Washington

[1] A. KOLMOGOROV, "Sulla determinazione empirica di una legge di distribuzione," Inst. Ital. Attuari, *Giorn.*, v. 4, 1933, p. 83–91.

[2] N. SMIRNOV, "On the estimation of the discrepancy between empirical curves of distribution for two independent samples," *Bull. Math. de l'Univ. de Moscou*, v. 2, 1939, fasc. 2. 16 p. The table referred to was also reproduced in *Annals Math. Stat.*, v. 19, 1948, p. 279–281.

1079[K].—F. J. MASSEY, JR., "Distribution table for the deviation between two sample cumulatives," *Annals Math. Stat.*, v. 23, 1952, p. 435–441.

Given two ordered samples of sizes N_1 and N_2 from cumulatives $F_1(x)$ and $F_2(y)$. $S_{N_i}(x)$ is the proportion of the members of the sample N_i $(i = 1, 2)$ less than x. Let $d = \max |S_{N_1}(x) - S_{N_2}(x)|$. Table 1 (p. 436–439) presents all possible values of d_α and α, for $N_1, N_2 \leqq 10$, where

$$\alpha = \text{Prob}\,[d \leqq d_\alpha \,|\, F_1(x) = F_2(x)].$$

The table is in terms of N_1, N_2, h, and α; h is an integer, such that $d_\alpha = h/l$, where l is the lowest common multiple of N_1 and N_2 $(h \leqq l)$; and α is given to 5D.

Table 2 (p. 440–441) is the same for additional selected values of N_1 and N_2 except that h usually is not carried out to l.

R. L. ANDERSON

University of North Carolina
Raleigh, North Carolina

1080[K].—J. M. MAY, "Extended and corrected tables of the upper percentage points of the 'Studentized' range," *Biometrika*, v. 39, 1952, p. 192–193.

This paper contains tables of the upper percentage points of the 'Studentized' range $q = (x_n - x_1)/s$ for the 5% and the 1% levels of significance for $n = 2(1)20$ and $\nu = 1(1)20, 24, 30, 40, 60, 120, \infty$ where s^2 is a mean square estimate of σ^2 independent of the range, n is the sample size for the range, and ν is the number of degrees of freedom of s. Most entries are given to 2D. None contains fewer than 3S. These tables were computed by numerical quadrature as a supplement to tables computed earlier by PEARSON & HARTLEY[1] with the aid of approximation formulas which were found unreliable for small ν and large q. The Pearson-Hartley tables did not include entries for $\nu < 10$ and the authors pointed out that entries exceeding 6 should not be considered reliable for $\nu \geqq 10$. For additional tables of the 'Studentized' range see the review of PILLAI's paper in this issue [**RMT 1083**].

A. C. COHEN, JR.

University of Georgia
Athens, Georgia

[1] E. S. PEARSON & H. O. HARTLEY, "Tables of the probability integral of the 'Studentized' range," *Biometrika*, v. 33, 1943, p. 89–99.

1081[K].—J. MOSHMAN, "Testing a straggler mean in a two-way classification using the range," *Annals Math. Stat.*, v. 23, 1952, p. 126–132.

In 1948 NAIR[1] obtained the distribution of the Studentized extreme deviate from the sample mean, i.e., he obtained the distribution of the sample statistic

$$u_\nu = (x_k - \bar{x})/s_\nu \qquad \text{or} \qquad u_\nu' = (\bar{x} - x_1)/s_\nu$$

where $x_1 \leqq x_2 \leqq \cdots \leqq x_k$, $\bar{x} = k^{-1} \sum_{i=1}^{k} x_i$ and s_ν is an independent estimate of the standard deviation based on ν degrees of freedom. In a two-way classification in the analysis of variance, therefore, s_ν can be taken as the residual standard deviation and the above test can be used to judge whether a row or column mean is high or low relative to the grand mean. In 1950 PATNAIK[2] published a paper which indicated that the distribution of the mean range can be approximated with sufficient practical accuracy by a χ^2 distribution, provided an appropriate scale factor is used and an equivalent (usually fractional) number of degrees of freedom employed in the conversion. Tables were published in Patnaik's article which give both the appropriate scale factor and the equivalent number of degrees of freedom for the mean range. These two results have apparently led the author of the present paper to recommend a test for detecting a "straggler" row or column mean in a two-way classification using in effect the mean range instead of the standard deviation in NAIR's test. As a matter of fact, the test proposed by MOSHMAN is based on the sample statistic

$$g = \frac{\left| k \sum_{i=1}^{r} x_{ik} - \sum_{i=1}^{r} \sum_{j=1}^{k} x_{ij} \right|}{W_{s,1}} \qquad \text{or} \qquad \frac{\left| r \sum_{j=1}^{k} x_{rj} - \sum_{i=1}^{r} \sum_{j=1}^{k} x_{ij} \right|}{W_{s,1}}.$$

The numerator of the g statistic hence involves only row or column totals, the grand total, the number of rows and columns and a function $W_{k,r}$ of certain "ranges." This latter factor is in fact the sum of the "column" ("row") ranges of the "row" ("column") residuals. Appropriate tables for the 1% and 5% significance levels of the statistic, g, are given to 3S in the article for all cases where the number of rows or columns does not exceed nine. A quick answer concerning whether a particular row or column mean in a two-way classification is a straggler can therefore be obtained directly from these tables. For another method see TUKEY.[3]

F. E. GRUBBS

Ballistic Research Laboratories
Aberdeen Proving Grounds, Maryland

[1] K. R. NAIR, "The distribution of the extreme deviate from the sample mean and its Studentized form," *Biometrika*, v. 35, 1948, p. 118–144.
[2] P. B. PATNAIK, "The use of the mean range as an estimator of variance in statistical tests," *Biometrika*, v. 37, 1950, p. 78–87.
[3] J. W. TUKEY, "Comparing individual means in the analysis of variance," *Biometrics*, v. 5, 1949, p. 99–114.

1082[K].—D. N. NANDA, "Probability distribution tables of the larger root of a determinantal equation with two roots," Indian Soc. Agr. Stat., *Jn.*, v. 3, 1951, p. 175–177.

Define the distribution function $P(x)$ by

$$2P(x)I(1, 2m + 2, 2n + 2) = 2I(x, 2m + 2, 2n + 2) - x^{m+1}(1 - x)^{n+1}I(x, m + 1, n + 1)$$

where

$$I(x, p + 1, q + 1) = \int_0^x u^p(1 - u)^q du.$$

The author tabulates $P^{-1}(.95)$ and $P^{-1}(.99)$ to 2D for $m = 0(\frac{1}{2})2, n = \frac{1}{2}(\frac{1}{2})10$. These tables provide tests for a number of statistical hypotheses, including that of the equality of two covariance matrices of bivariate normal populations.

J. L. HODGES, JR.

University of California
Berkeley, California

1083[K].—K. C. S. PILLAI, "On the distribution of the 'Studentized' range," *Biometrika*, v. 39, 1952, p. 194–195.

This paper contains tables of the lower percentage points of the 'Studentized' range $q = (x_n - x_1)/s$ at the 5% and the 1% levels of significance for $n = 2(1)8$ and $\nu = 1(1)9$, where the notation is that of RMT **1080**. All entries are given to 2D. These results further supplement those of PEARSON and HARTLEY.[1] An editorial note explains that PILLAI also computed certain upper percentage points which were not published separately since his paper was submitted while MAY's (see RMT **1080**) more extensive program of computation was in progress. Pillai's upper 5% and 1% points computed for $1 \leqq \nu \leqq 4$ and $2 \leqq n \leqq 8$ are included in the tables by May, and according to May, agreed with his calculations to within a unit of the last figure quoted by Pillai. In order to perform the necessary calculations, the

author determined the distribution of q explicitly in the form of an infinite series.

A. C. COHEN, JR.

University of Georgia
Athens, Georgia

[1] E. S. PEARSON & H. O. HARTLEY, "Tables of the probability integral of the 'Studentized' range," *Biometrika*, v. 33, 1943, p. 89–99.

1084[K].—ALAN STUART, "The power of two difference-sign tests," Am. Stat. Assn., *Jn.*, v. 47, 1952, p. 416–424.

The author considers a distribution-free test of serial independence of N (unequal) observations with its $N-1$ first differences ordered in time, a test proposed by MOORE & WALLIS.[1]

Define $D = \sum_{i=1}^{N-1} D_i$ where D_i is unity when the ith difference is positive, and zero otherwise. The null hypothesis is that the observations come from the same continuous population, and under this hypothesis the mean and variance of D is derived and its asymptotic normality for large N proved. The alternative hypothesis, assuming the observations equally spaced, is $y_k = \alpha + \beta k + x_k$ where x_k is a normal variate with mean β and variance σ^2. The mean, variance, and the asymptotic normality under the alternative hypothesis is derived. Assuming exact normality of the statistic D under both the null and alternative hypotheses, the author tabulates the power of the D-test to 2D at the 95% level of significance for $N = 15, 25, 50, 75, 100$ and $R = \beta/\sigma\sqrt{2} = .1(.1).5, 1$. He concludes for $N \geqq 25$, $R \geqq .5$, and $N \geqq 75$, $R \geqq .3$, that the loss of power as compared with the best available test is at most 8 percent. He next considers the case of two series of N unequal observations in time, and to investigate the correlation between them defines D_i for the first series as before, and similarly d_i for the second series. Let $C_i = D_i d_i$, and the correlation statistic $C = \sum_{i=1}^{N-1} C_i$. Under the null hypothesis that the N pairs of observations are drawn from the same continuous bivariate population in which the two variates are independent he finds the mean variance of C and proves its asymptotic normality. The alternative hypothesis is that the N pairs of observations are drawn from a normal bivariate population with $\rho \neq 0$, and under this hypothesis the mean, variance of C and its asymptotic normality is derived. The author provides a table for the power of the C test to 2D against this alternative (under the assumption again of exact normality) at the 95% level for $\rho = .1(.1).7$ and $N = 50, 100, 200$, and 400. This is compared to the power of the usual Z test, $Z = \frac{1}{2} \log \left(\frac{1+r}{1-r} \right)$ at the 95% level for $\rho = .1(.1).5$, $N = 50, 100, 200, 400$, with results given to 2D. For $\rho \geqq .5$, $N \geqq 100$, $\rho \geqq .4$, $N \geqq 200$; or $\rho \geqq .3$, $N \geqq 400$, he concludes that the use of the C test is satisfactory since at most 8% of the power is lost.

L. A. AROIAN

Hughes Aircraft Company
Culver City, California

[1] G. H. MOORE & W. A. WALLIS, "Time series significance tests based on signs of differences," Amer. Stat. Assn., *Jn.*, v. 38, 1943, p. 153–164.

1085[K].—T. J. TERPSTRA, "A confidence interval for the probability that a normally distributed variable exceeds a given value, based on the mean and the mean range of a number of samples," *Applied Scientific Research*, A, v. 3, 1952, p. 297–307.

The author employs PATNAIK's[1] method to find the confidence intervals described in the title. For l samples of size k each the author gives four charts from which one can read the central 90% confidence limits on the vertical axis with the ratio of the numerical difference of the given value from the mean of the total sample to the average range on the horizontal axis, for $l = 4(2)10$ and $k = 1(1)5$.

C. C. C.

[1] P. B. PATNAIK, "The use of the mean range as an estimator of variance in statistical tests," *Biometrika*, v. 37, 1950, p. 78–87.

1086[K].—J. WISHART, "Moment coefficients of the k-statistics in samples from a finite population," *Biometrika*, v. 39, 1952, p. 1–13.

TUKEY[1] in order to simplify the problem of finding moment coefficients of Fisher's k-statistics in samples from a finite population introduced the sample statistic $K_{r_1 \ldots}$ which has the property that its expected value is the same function of the N members of the population that $k_{r_1 \ldots}$ is of the n members of the sample. The author first gives a conversion table for finding the $k_{r_1 \ldots}$ statistics in terms of the augmented monomial symmetric functions of DAVID & KENDALL[2] and vice versa through order 6. Then to facilitate the derivation of further sampling formulae he gives a complete table of products and powers of k-statistics in terms of the $k_{r_1 \ldots}$ through order 6. For orders 7 and 8 he gives the 14 and 21 basic formulae respectively, the remaining results of those weights then being derivable by algebraic substitution. Illustrations of the application of these tables are given.

C. C. C.

[1] J. W. TUKEY, "Some sampling simplified," Amer. Stat. Assn., *Jn.*, v. 45, 1950, p. 501–519.
[2] F. N. DAVID & M. G. KENDALL, "Tables of symmetric functions—part I," *Biometrika*, v. 36, 1949, p. 431–449. [*MTAC*, v. 4, p. 146.]

1087[L].—BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Committee on Mathematical Tables, *Table of Bessel Function*, Part II. Cambridge, 1952. xl + 255 p., 21.5 × 28 cm.

This is the long-awaited "second volume." The functions tabulated are $J_n(x)$, $Y_n(x)$, $I_n(x) = i^{-n}J_n(ix)$ and $K_n(x) = \frac{1}{2}\pi i^{n+1}[J_n(ix) + iY_n(ix)]$ for $n = 2(1)20$ (the first volume[1] gave values of the functions for $n = 0, 1$) and for $x = 0(.1 \text{ or } .01)10(.1)20$ with δ^2 or δ_m^2. Tables for J_n are 8D whereas those for Y_n, I_n, and K_n are to 8 figures. In regions where Y, I, or K is not convenient for interpolation, the functions x^nY_n, x^nK_n, $e^{-x}I_n$ or e^xK_n are tabulated. Finally J_n, Y_n, I_n, and K_n are tabulated for $n = 0(1)20$, $x = 0(.1)25$, to 10D for J_n, 10 figures for Y_n, I_n, K_n.

The tables are, in part, based on those of ALDIS[2] but are very considerably expanded and reworked. The checking has been carried out with the scrupulousness typical of this series of tables.

Bessel functions are like the weather, they nearly always intrude somewhere in any research in theoretical physics or applied mathematics. Like the weather also, for a long time very little was done about assembling a set of tables of the functions complete enough to satisfy most needs. Few of the calculations of acoustical and electromagnetic wave radiation, propagation or scattering can be made with the sole use of Bessel functions of the first kind, of real argument, so attempts to use the remarkable HARVARD COMPUTATION LABORATORY[3] tables are usually reminiscent of attempts to skate with only one ice-skate. Here, in the present BAASMTC volume, we have *both* solutions tabulated, for both real and imaginary values of the argument, with enough significant figures and enough range of the argument and order so that most calculations can be carried through. The Bessel function Panel responsible for its production—W. G. BICKLEY, L. J. COMRIE, J. C. P. MILLER, D. H. SADLER, and A. J. THOMPSON—are to be congratulated. Incidentally, this is the last volume of the BAAS tables; the Committee has been reconstituted as the Royal Society Mathematical Tables Committee.

As far as the needs of the theoretical physicist and engineer are concerned, this volume and the earlier one[1] will suffice, in the majority of cases. In a few scattering calculations the functions J and Y for $n = 20(1)50$ for $x = 0(.1)25$ would be useful, for in scattering calculations, orders up to about twice the argument, still contribute in the result. In some electromagnetic and acoustical problems, functions of a complex argument are required. These are available in the NYMTP tables[4] for $n = 0$ and 1. It would be useful to have similar tables for $n = 2(1)10$, but such needs are not urgent. Tables available and desired for half-integer orders will be discussed in RMT 1092.

The volume under review here is clearly printed and each page is clearly marked for ready reference. It is a pleasure to use the volume.

PHILIP M. MORSE

Massachusetts Institute of Technology
Cambridge 39, Mass.

[1] BAASMTC, *Bessel Functions*, Part I. Cambridge, 1937, xx + 288 p.
[2] W. S. ALDIS, Roy. Soc. *Proc.*, v. 64, p. 203, 1899, and v. 66, p. 32, 1900.
[3] HARVARD COMPUTATION LABORATORY, *Tables of Bessel Functions of the First Kind for* $n = 0$ to 135, vols. III to XIV, 1947–1951, Harvard U. Press, Cambridge, Mass.
[4] NYMTP, *Tables of* J_0, J_1 *and* Y_0, Y_1 *for Complex Arguments*, 1943 and 1950, Columbia University Press, New York.

1088[L].—H. KOBER, *Dictionary of Conformal Representations*. New York, Dover, 1952. xvi + 208 p., 15.2 × 23.8 cm. Price $3.95.

The book represents a formidable achievement in bringing together a wide variety of explicit conformal maps useful to physicists, engineers and computers. Particularly notable are the numerous clear diagrams, which allow the reader to locate quickly a required mapping. Most of the transformations are by elementary functions, but short theoretical discussions of the Schwarz-Christoffel transformation and of the Schwarz principle of reflection indicate in part the derivation of more complicated formulas. Proofs are altogether omitted due to the dictionary character of the presentation. Mappings are classified according to the nature of the analytic func-

tions involved, with chapters on bilinear transformations, algebraic functions, trigonometric and exponential functions, the Schwarz-Christoffel transformation, and elliptic and modular functions. Although a geometrical index is listed as topological, almost no reference is made to Riemann surfaces. This is particularly to be regretted, since it is quite common in engineering problems, such as those involving the hodograph method, to ignore the usefulness and necessity of exploiting simple conformal mappings onto Riemann surfaces. Accuracy and a notable absence of misprints, of which the reviewer found only one, should make the handbook of permanent and growing value to the applied scientist.

P. R. GARABEDIAN

Stanford University
California

[*Editorial Note.* A preliminary version of this work was reviewed in *MTAC*, v. 2, p. 296–297, v. 3, p. 103, 368].

1089[L].—N. H. MARCH, "Thomas-Fermi fields for molecules with tetrahedral and octahedral symmetry," Cambridge Phil. Soc., *Proc.*, v. 48, 1952, p. 665–682.

Two types of solutions of the differential equation

$$\frac{d^2\phi}{dx^2} = x^{-\frac{1}{2}}\phi^{\frac{1}{2}}$$

are tabulated in this paper. One of these is of the form

$$\phi = 1 + a_2x + a_3x^{\frac{3}{2}} + a_4x^2 + \cdots$$

and is fully determined once a_2 is given. Table 2 (p. 670) gives $a_3, \cdots, a_{11}$ in terms of a_2, and Table 1 (p. 668) values of ϕ for $-a_2 = 1.55, 1.56, 1.567, 1.5748, 1.581, 1.5849, 1.5853, 1.5866, 1.58756, 1.58764$. x varies from 0 to 10.74. The solutions were started by using the above series and then continued by numerical integration of the differential equation. Table 1 also gives in each case the point x_0 the tangent at which passes through the origin, with the corresponding $\phi(x_0)$. It is suggested that a classification of the solutions by $\phi(x_0)$ is more reliable than that based on a_2.

The second type of solutions is of the form

$$\phi = \frac{144}{x^3}\left(1 - \frac{F_1}{x^c} + \frac{F_2}{x^{2c}} - \frac{F_3}{x^{3c}} + \cdots\right).$$

Table 3 (p. 670) gives expressions of $F_2, \cdots, F_{10}$ in terms of F_1. It turns out that all solutions of this type may be derived from two "master solutions" in which F_1 is, respectively, 1 or -1. These two master solutions and their first derivatives are tabulated in Tables 4 and 5 (p. 671, 672) for values of x which range from .1375 to 20.2 in Table 4 ($F_1 = 1$), and from .5375 to 19.4 in Table 5 ($F_2 = -1$).

In the paper there are also references to solutions computed previously.[1]

A. E.

[1] R. P. FEYNMAN, N. METROPOLIS, & E. TELLER, "Equations of state of elements based on the generalized Fermi-Thomas Theory," *Phys. Rev.*, s. 2, v. 75, 1949, p. 1561.
J. C. SLATER & H. M. KRUTTER, "The Thomas-Fermi method for metals," *Phys. Rev.*, s. 2, v. 47, 1935, p. 559.

1090[L, M].—W. MEYER ZUR CAPELLEN, *Integraltafeln. Sammlung unbestimmter Integrale elementarer Funktionen.* Springer-Verlag. Berlin/Göttingen/Heidelberg, 1950. viii + 292 p., 23.8 × 16.5 cm.

This book contains probably the largest published collection of indefinite integrals whose integrands are combinations of elementary functions. It was undertaken in 1942 and completed in 1944, publication being delayed by difficulties of the post-war period. There are about 3000 entries, most of them giving the final answer, and some of them stating recurrence relations or other auxiliary formulae for the integrals in question. Many special cases of more general results were included, thus enabling the user to get the final answer quickly. For instance, in section 2.2.2, there are recurrence relations for the integral

$$\int x^p(a + bx)^{n/m}dx \qquad m, n, p \text{ integers},$$

and there is also an explicit formula for positive integer p. Nevertheless, there are about 7 pages of some eighty particular cases listed, for special values of m, n, p—a great convenience for the user.

Contents. Chap. 1. Preliminaries (5 p.). Rules for differentiation and integration. Chap. 2. Integrals of algebraic functions (138 p.). 2.1. Rational integrands. 2.2. Irrational integrands leading to elementary functions. 2.3. Algebraic integrands leading to elliptic integrals. Chap. 3. Integrals of transcendental functions (80 p.). 3.1. Exponential and logarithmic functions. 3.2. Trigonometric and inverse trigonometric functions. 3.3. Hyperbolic and inverse hyperbolic functions. Chap. 4. Products of algebraic and transcendental functions (45 p.). Arranged according to the transcendental part, similar subdivision to that of Chapter 3. Chap. 5. Products of transcendental functions (11 p.). 5.1. $\int g(x) \ln x dx$. 5.2. $\int e^x g(x)dx$. Addenda (3 p.). Chap. 6. Supplementary material (9 p.). Constants, expansions, definitions of some higher transcendental functions. Chap. 7. References (19 items, 1 p.).

The classification and arrangement of such a large number of integrals presents very formidable difficulties, and only continued and frequent use of the tables will show the excellence, or the imperfections, of the scheme adopted here. The author's aim was to provide tables which can be used with ease by a non-mathematician, and the first impression is decidedly favorable. There is a very detailed table of contents which enables one to locate any integral printed in the book within four or five pages. One will, of course, encounter integrals which are not in the book, yet can be transformed into integrals printed in the book; but a generous duplication of several forms of the same integral will greatly reduce the number of these occurrences.

The only drawback the reviewer found in using these tables is the absence of a list of notations and abbreviations. The user not acquainted with German books will find out sooner or later that Sin means sinh and ArSin means $\sinh^{-1}$ (although these symbols are defined on p. 286, 287 under hyperbolic functions, meeting them on p. 77 as integrals of irrational functions gives no clue as to the place where one should look for their definition). Ad hoc notations are sometimes explained where they occur (for instance, x on p. 63 in 2.3.1.2.1.3.1) or in a footnote on the same page on which they

are used (as U, V, W, on p. 68); yet sometimes one has to turn several pages until one finds an explanation (the same V, W, are used on p. 76, and one has to go back to p. 68 to find a footnote referring one to nos. 2.2.1.1.1 and 2.2.1.1.2 which are then found on p. 73, 74). Also, the numbering of the entries (samples above) is cumbersome, if systematic, and makes it inconvenient to give specific references.

In all other respects this is an eminently useful book, and the printing is excellent.

A. E.

[For errata in this work see in this issue MTE 223.]

1091[L].—NBS Applied Mathematics Series, No. 17. *Tables of Coulomb wave functions.* U. S. Government Printing Office, Washington, D. C., 1952. xxvii + 141 p., 20.5 × 27 cm. Price, $2.00.

Coulomb wave functions are confluent hypergeometric functions, but the normalisation and notation usual in quantum mechanics differ from the standard notations in mathematical texts.

The "regular Coulomb wave function," $F_L(\eta, \rho)$ is that solution of the differential equation

$$\frac{d^2y}{d\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right] y = 0$$

which is regular at $\rho = 0$ and is so normalised that

$$F_L(\eta, \rho) \sim \sin(\rho - \eta \ln 2\rho - \tfrac{1}{2}L\pi + \sigma_L)$$

as $\rho \to \infty$, where $\sigma_L = \arg \Gamma(L + 1 + i\eta)$. The notation

$$F_L(\eta, \rho) = C_L(\eta)\rho^{L+1}\Phi_L(\eta, \rho)$$

is frequently used, where

$$C_0(\eta) = [2\pi\eta/(e^{2\pi\eta} - 1)]^{\frac{1}{2}}, \quad L(2L+1)C_L(\eta) = (L^2 + \eta^2)^{\frac{1}{2}}C_{L-1}(\eta),$$

and $\Phi_L(\eta, \rho)$ is a power series in ρ.

Coulomb wave functions have been computed by several authors: on p. 152 of the recent tables by Bloch *et al* (*MTAC*, v. 6, p. 92), there are references to available tables. The volume under review contains the most extensive, and the most systematic, tables, and it largely (but not entirely) supersedes the earlier tabulations.

Table I (p. 1–111) gives 7D values of $\Phi_L(\eta, \rho)$, and of as many reduced derivatives

$$\frac{1}{k!}\frac{d^k}{d\eta^k}\Phi_L(\eta, \rho)$$

as are needed for interpolation. In this table, $\eta = -5(1)5$, $L = 0(1)5$, 10, 11, 20, 21, and $\rho = 0(.2)5$.

Table II (p. 113–127) gives 10D values of the real part of $\Gamma'(1 + i\eta)/\Gamma(1 + i\eta)$, with modified second central differences, for $\eta = 0(.005)2(.01)6(.02)10(.1)20(.2)60(.5)110$.

Table III (p. 129–135) gives 8D values of σ_θ, with modified second central differences, for $\eta = 0(.01)1(.02)3(.05)10(.2)20(.4)30(.5)85$.

Table IV (p. 137–141) gives 8S values of $C_0(\eta)$ for $\eta = 0(.01)5(.05)10$.

The Foreword contains an account of the applications of Coulomb wave functions, and the Introduction (by M. ABRAMOWITZ) gives definitions of, and formulas, Bessel function expansions, and asymptotic expansions for, Coulomb wave functions. The Introduction also describes the method of computation, and gives instructions for interpolation and for the use of recurrence relations.

In view of the fact that this volume deserves, and is likely, to become the principal work of reference for Coulomb wave functions, without entirely superseding some of the earlier tables, it is regrettable that there is no systematic account of all (although there are references to some) available tables.

A. E.

1092[L].—ROYAL SOCIETY MATHEMATICAL TABLES COMMITTEE, *Short Table of Bessel Functions* $I_{n+\frac{1}{2}}$, $K_{n+\frac{1}{2}}$. Cambridge, 1952, 20 p., 21.5 × 28 cm.

This is a preliminary table of spherical Bessel functions of imaginary argument. The functions actually tabulated are

$x^{-n-1}I_{n+\frac{1}{2}}(x)$	for	$n = 0(1)10$	and	$x = 0(.1)5$,
$(2/\pi)x^{n+1}K_{n+\frac{1}{2}}(x)$	for	$n = 0(1)10$	and	$x = 0(.1)5$,
$e^{-x}I_{n+\frac{1}{2}}(x)$	for	$n = 0(1)10$	and	$x = 5(.1)10$,
$(2/\pi)e^{x}K_{n+\frac{1}{2}}(x)$	for	$n = 0(1)10$	and	$x = 5(.1)10$,

in most cases to eight significant figures, together with δ_m^2, for interpolation by Everett's formula, when $I_n(x) = i^{-n}J_n(ix)$ and

$$(2/\pi)K_{n+\frac{1}{2}}(x) = (-1)^{n+1}[I_{n+\frac{1}{2}}(x) - I_{-n-\frac{1}{2}}(x)].$$

These tables begin the completion of the spherical Bessel functions as the table in RMT 1087 completes the main tabulation for cylindrical Bessel functions. The NBS tables[1] have covered the two independent solutions, for real values of the argument, for $n = 0(1)30$ and $x = 0(.01$ or $.1)25$ to 8 or 10 significant figures. The present table is a start toward a similar table for imaginary arguments.

Spherical Bessel functions of imaginary argument are useful in the quantum mechanical calculation of atomic and molecular properties. In particular, an atomic wave function for a bound state, about one center, when expanded about another center, appears as a sum of such Bessel functions, multiplied by spherical harmonics.[2] Calculations of interatomic forces in a molecule have been quite difficult to carry out, chiefly because of the lack of adequate tables. Now, with the advent of high speed computing machines, to remove the burden of routine details, such computations should become relatively easy to perform and our insight into molecular structure can begin to be satisfactorily complete. Values of $I_{n+\frac{1}{2}}(x)$ and $K_{n+\frac{1}{2}}(x)$ for $n \simeq 0(1)20$ and $x \simeq 0(.1)20$ will be needed as a basis for the calculation of the component integrals. The present table is a promising

start in this direction; let us hope that the RSMTC, or some other group, will carry on.

PHILIP M. MORSE

Massachusetts Institute of Technology
Cambridge 39, Mass.

[1] NBSCL, *Tables of Spherical Bessel Functions*. Vols. I and II, 1947, Columbia University Press, New York.

[2] See, for example, N. ROSEN, "Interaction between Atoms with s-Electrons," Phys. Rev., v. 38, p. 255, 1931.

1093[V].—AERONAUTICAL RESEARCH COUNCIL, *A Selection of Tables for Use in Calculations of Compressible Airflow*. Prepared by the Compressible Flow Tables Panel (L. ROSENHEAD, Chairman; W. G. BICKLEY, C. W. JONES, L. F. NICHOLSON, C. K. THORNHILL, R. C. TOMLINSON) of the ARC. Oxford, 1952, viii + 143 p., 21.5 × 27.0 cm. Price $8.00.

The earlier and more familiar tables for compressible flow in this country are H. W. EMMONS' "Gas Dynamics Tables for Air" (Dover publications, 1947, *MTAC*, v. 3, p. 36) and J. H. KEENAN & J. KAYE'S "Gas Tables" (Wiley, 1945, *MTAC*, v. 2, p. 20). Compared with these tables, the present publication is certainly more detailed and more accurate. It is stated that the two earlier tables have numerous errors, mostly affecting the final place of decimals. As in Emmons' table, the value of the ratio of specific heats, γ, is chosen to be 1.4. The authors try to remedy this limitation in part by listing the derivatives of a few important quantities with respect to γ at $\gamma = 1.4$. But nevertheless, this restriction will make the table unsuitable for calculations of gas flow at higher temperatures, such as the flow in high pressure compressors and gas turbines.

There are three main groups of tables: The first group is concerned with isentropic one-dimensional flow. The argument used is the local Mach number M, the ratio S of speed to the critical sonic speed, or the square of the ratio of speed to maximum speed τ. The M-tables have the ranges $M = 0(.01).5(.001)1; 1(.01)5$. The S-tables have the ranges $S = 0(.01)2.44$ and $\sqrt{6}; 1(.005)1.5(.01)2.44$ and $\sqrt{6}$. The τ-table has the range $\tau = 0(.01)1$.

The second group of tables is for the solution by the characteristics method. The argument used is either the angular characteristics variable ω or the Mach angle μ. The ranges of these tables are $\omega = 0°(.5°)100°(1°)130°; 0°(.001°).05°; 0°(.01°)1°, 0°(.1°)30°$. $\mu = 0°(1°)90°$.

The third main group of tables gives the ratio of quantities after the shock to that before the shock. The argument of the tables is either the Mach number M_1 or the speed ratio S_1 before the shock. For normal shock waves, $M_1 = 1(.01)5$; $S_1 = 1(.01)2.44$ and $\sqrt{6}$. For oblique shock waves only the maximum flow deflection angle and angle for sonic flow after shock are given for $M_1 = 1(.01)2(.1)5(1)20$.

Other tables in this volume are tables for obtaining Mach number from the dynamic pressure ratio or the Pitot pressure ratio, tables of pertinent powers of x and $(1 - x^2)$, tables for calculating the Reynolds number, the pressure coefficient, etc.

H. S. TSIEN

California Institute of Technology
Pasadena, California

MATHEMATICAL TABLES—ERRATA

In this issue references to errata have been made in RMT's **1065**, **1068** and **1069**.

221.—L. A. LĬŪSTERNIK, I. Ĭa. AKUSHSKIĬ, & V. A. DITKIN, *Tablitsy Besselevykh Funktsiĭ*. (*Matematicheskie Tablitsy*, no. 1.) [Tables of Bessel Functions (Mathematical Tables, v. 1)] 1949. [*MTAC*, v. 5, p. 25.]

In addition to the errors noted in *MTAC*, v. 5, p. 25, the following errata were discovered by inspection and checking of differences.

	x	$J_0(x)$		$J_1(x)$	
p.		*for*	*read*	*for*	*read*
11	0.167	9920399	9930399		
25	0.860			3914929	3914529
41	1.678			694	674
46	1.944	2592767	2562767		
51	2.171	1265437	1265435		
54	2.343	5119	5319		
67	2.983			3457353	3453753
74	3.345			2000896	2020896
81	3.662	608	698		
86	3.939			425404	425434
87	3.960			506903	506953
89	4.099	8887728	3887728		
94	4.316	4769	1769		
94	4.326			0801626	1801626
143	6.755	791	789		
148	7.046			−0.0091064	+0.0091064
162	7.750			1916006	1916026
170	8.141			2501847	2521847
181	8.674			2707632	2707622
204	9.849			2247	2447
205	9.896	696	694		
208	10.028			2606	2506
231	11.179	2914	2014		
254	12.350			1356	1354
266	12.930	2040869	2014869		
271	13.152	2151370	2151350		
274	13.340			25587	35587
279	13.575	2115728	2115428		
286	13.908			1179283	1179223
294	14.317	1210012	1215012		
299	14.557	760050	764050		
299	14.562	750175	754175		
299	14.577	720459	724459		
304	14.837	190310	194310		
308	15.047	230301	238301		

	x	$J_0(x)$		$J_1(x)$	
p.		*for*	*read*	*for*	*read*
310	15.128			1998981	1998921
311	15.173			1972955	1972925
328	16.260	1920047	1921047		
332	18.490			162	164
345	25.000	101	99		

Three dozen values in which the errors amounted to one unit in the last place have not been included in this list.

S. A. JOFFE

515 W. 110 Street
New York 25, N. Y.

222.—N. W. MCLACHLAN, P. HUMBERT, & L. POLI, *Supplément au Formulaire pour le Calcul Symbolique.* Mémorial des Sciences Mathématiques, fasc. 113, 1950.

In addition to the errata published in *MTAC*, v. 6, p. 101, the following errors occur.

p. 5, item 2; *for* Ei$(-x)$ on the r.h.s. *read* Ei(x).
p. 5, line 10; *insert*) after x.
p. 6, formula 7; *for* the last $+$ *read* $-$.
p. 7, formula 3; *for* x^a *read* x^{-a}.
p. 10, second item for p. 12; *for* $\phi(\log p)$ *read* $\phi(\log p) - f(0)$.
p. 10, second item for p. 14; *for* -1 *read* $-p$.
p. 11, item for p. 34; *read* delete formulas 3–10.
p. 11, item for p. 47; *for* $-a^3/s^3$ *read* $-a^2/s^3$.
p. 11, item for p. 48; *for* $I_n(p^2/2)$ *read* $I_n(2/p^2)$ as in first edition.
p. 14, formula 10; *for* p^n *read* p.
p. 15, formula 8; lower limit in second integral should be p.
p. 16, formula 8; *for* 2^n *read* $2^{n/2}$, and *for* $x/2\sqrt{t}$ *read* $x/\sqrt{2t}$.
p. 22, sixth formula from foot; in denominator *for* p *read* p^ν.
p. 26, formulas 4, 5; *delete*.
p. 27, last formula; *replace* l.h.s. *by* $\log\{(t+1)/(t-1)\}$, and r.h.s. *by* $e^{-p}(\log 2p + \gamma) - e^p \operatorname{Ei}(-2p)$, $t > 1$.
p. 28, formula 1; *multiply* the r.h.s. *by* 2.
p. 31, formulas 3, 8; the centre sign on the l.h.s. should be $+$.
p. 43, formula 9; on the r.h.s. *for* $(ab)^{2m+\nu}(2b)^{\mu+1}$ *read* $2^{\mu+1}a^{2m+\nu}b^{\frac{1}{2}(\mu+\nu+1)+m}$.
p. 43, in the last two integrals, *for* dx *read* dt.

I am indebted to A. ERDÉLYI for some of these corrections.

N. W. MCLACHLAN

Vizard & Co.
51 Lincoln's Inn Fields
London W.C.2, England

223.—W. MEYER ZUR CAPELLEN, *Integraltafeln*, Springer, Berlin, 1950 [RMT 1090].

P. 224, column 3, formula 1.1.0.1, line 3. *For* $(n - 3)$ *read* $(n - 2)$.

P. 274, column 3, formula 1.3.1.2, line 2.

For

$$U(x) = ax^2 - \frac{2a^2 - b^2}{p}x - \frac{2a}{p^2}(a^2 - 3b^2)$$

read

$$U(x) = ax^2 - \frac{2(a^2 - b^2)}{p}x + \frac{2a}{p^2}(a^2 - 3b^2).$$

P. 275, column 3, formula 2.3.1.0. *For* $(a \sin bx + b \cos bx)$ *read* $(a \cos bx + b \sin bx)$.

These are in addition to the loose leaf sheet of errata supplied with the book.

MURLAN S. CORRINGTON

Radio Corporation of America
Camden, New Jersey

224.—NBSMTP, *Table of Natural Logarithms*, v. 4. New York, 1941.

Table I, p. 484, in the argument column

for $x = 9.2834$ *read* $x = 9.8234$

YVONNE SETTLE

Midwest Research Inst.
Kansas City, Missouri

225.—J. W. NICHOLSON & J. R. AIREY, BAAS *Report*, 1927.

In the table of confluent hypergeometric functions, p. 220–254, there is an error on p. 243 in $M(7/2, 4, .5)$.

for $x = 1.55306$ *read* 1.55290

YVONNE SETTLE

Midwest Research Institute
Kansas City 2, Missouri

226.—G. W. SPENCELEY & R. M. SPENCELEY, *Smithsonian Elliptic Functions Tables*, Publ. no. 3863, Washington, 1947. [See *MTAC*, v. 3, p. 89.]

The whole $E(\phi, k)$ column (p. 71, 73) for the modular angle $\theta = 18°$ is wrong. It should read:

r	$E(\phi, k)$	r	$E(\phi, k)$
0	0.00000 00000 00		
1	0.01789 37526 82	6	0.10732 43169 64
2	0.03578 64115 08	7	0.12519 52058 38
3	0.05367 68841 57	8	0.14305 85232 78
4	0.07156 40813 75	9	0.16091 32180 25
5	0.08944 69184 96	10	0.17875 82508 48

r	$E(\phi, k)$	r	$E(\phi, k)$
11	0.19659 25959 83	51	0.89234 65066 55
12	0.21441 52425 30	52	0.90917 36431 51
13	0.23222 51958 40	53	0.92597 22079 17
14	0.25002 14788 57	54	0.94274 25147 28
15	0.26780 31334 34	55	0.95948 49102 74
16	0.28556 92216 21	56	0.97619 97735 85
17	0.30331 88268 98	57	0.99288 75154 29
18	0.32105 10553 91	58	1.00954 85776 81
19	0.33876 50370 29	59	1.02618 34326 61
20	0.35645 99266 71	60	1.04279 25824 46
21	0.37413 49051 83	61	1.05937 65581 55
22	0.39178 91804 68	62	1.07593 59192 16
23	0.40942 19884 56	63	1.09247 12526 02
24	0.42703 25940 37	64	1.10898 31720 49
25	0.44462 02919 51	65	1.12547 23172 55
26	0.46218 44076 22	66	1.14193 93530 55
27	0.47972 42979 52	67	1.15838 49685 84
28	0.49723 93520 44	68	1.17480 98764 16
29	0.51472 89918 93	69	1.19121 48116 95
30	0.53219 26730 08	70	1.20760 05312 41
31	0.54962 98849 91	71	1.22396 78126 53
32	0.56704 01520 56	72	1.24031 74533 91
33	0.58442 30334 98	73	1.25665 02698 50
34	0.60177 81241 05	74	1.27296 70964 18
35	0.61910 50545 24	75	1.28926 87845 34
36	0.63640 34915 61	76	1.30555 62017 22
37	0.65367 31384 42	77	1.32183 02306 31
38	0.67091 37350 12	78	1.33809 17680 52
39	0.68812 50578 87	79	1.35434 17239 46
40	0.70530 69205 52	80	1.37058 10204 46
41	0.72245 91734 12	81	1.38681 05908 66
42	0.73958 17037 93	82	1.40303 13787 01
43	0.75667 44358 87	83	1.41924 43366 20
44	0.77373 73306 65	84	1.43545 04254 61
45	0.79077 03857 25	85	1.45165 06132 12
46	0.80777 36351 12	86	1.46784 58740 01
47	0.82474 71490 80	87	1.48403 71870 72
48	0.84169 10338 19	88	1.50022 55357 67
49	0.85860 54311 38	89	1.51641 19065 02
50	0.87549 05181 05	90	1.53259 72877 46

G. W. SPENCELEY

402 E. Church St.
Oxford, Ohio

UNPUBLISHED MATHEMATICAL TABLES

156[A].—M. K. PAGE & M. K. PFEIL, *Further Computation of the Digits of e.* One page tabulated from punched cards. Deposited in the UMT FILE.

This gives the value of e to 3333D as obtained on the IBM 604 calculating punch. This work supplements that of F. GRUENBERGER and O. MARLOWE as reported in *MTAC*, v. 6, p. 123–124. The remainders of the various terms at 3333D and 3347D will be furnished upon request.

M. K. PAGE
M. K. PFEIL

Numerical Analysis Laboratory
University of Wisconsin
Madison, Wisconsin

157[A].—J. W. WRENCH, JR., Log π *and Related Values.* One typewritten page deposited in the UMT FILE.

The values of $\ln \pi$, $\log \pi$, $\frac{1}{2} \ln 2\pi$, $\frac{1}{2} \log 2\pi$, and $\log 2$ are given to 329D. The first of these was computed from

$$\ln \pi = \ln (2^{10} \cdot 3 \cdot 7^6 \cdot 17^5) - 4 \log (5^2 \cdot 11 \cdot 13) + \log (1 - \epsilon)$$

where ϵ is approximately $2.51869 \cdot 10^{-9}$. These values confirm the 214D values published by UHLER.[1]

J. W. WRENCH, JR.

4711 Davenport St., N.W.
Washington 16, D. C.

[1] H. S. UHLER, "Log π and other basic constants," Nat. Acad. Sci., Washington, *Proc.*, v. 24, 1938, p. 23–30.

158[F].—A. GLODEN, *Solutions of* $x^4 + 1 \equiv 0 \pmod{p}$ *for* $600000 < p < 800000$. Typewritten manuscript, 22 p. Deposited in the UMT FILE.

This is a continuation of two previous tables for $p < 600000$ [*MTAC*, v. 3, p. 96]. The present table gives two solutions for every possible prime p between 600000 and 800000.

With the help of these tables various results on the factors of $x^4 + 1$ were obtained, such as

$$820^4 + 1 = 626929 \cdot 721169.$$

These are appended.

A. GLODEN

11 rue Jean Jaurès
Luxembourg

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 415 South Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

THE ELECTRONIC COMPUTER AT THE INSTITUTE FOR ADVANCED STUDY

An all-electronic, general purpose, digital computer has been in operation at the Institute for Advanced Study in Princeton, New Jersey, since June 1952.

The overall characteristics of the Institute machine are displayed below and may be seen to follow closely the requirements established by BURKS, GOLDSTINE and VON NEUMANN in June 1946.

MACHINE CHARACTERISTICS

OVERALL

Binary
Parallel
40 bit words
2300 thermionic tubes
$2' \times 6' \times 8'$ in size
16 kilowatts of power

CONTROL

1-Address code with each 40 digit word containing two 20 digit orders
Asynchronous
Direct coupled
20 basic instructions
10 bit instructions

MEMORY

William's type electrostatic storage
1024 word capacity
40 parallel stages
Standard 5CP1A cathode ray tubes
Synchronous memory control
24 μs. period

ARITHMETIC

Completely direct coupled
Fixed binary point
Complemented representation of negative numbers
All numbers $-1 \leq x < +1$
12 μs. carry time
60 μs. addition order time
700 μs. average multiplication order time
1100 μs. division order time

INPUT–OUTPUT

IBM PUNCHED CARD

12 words per card
40 binary digits per word
100 cards per minute—load
100 cards per minute—punch

TELETYPE TAPE

5 holes across
4 bits information
8 minutes per 1024 words—load
32 minutes per 1024 words—print
16 minutes per 1024 words—punch

MAGNETIC DRUM

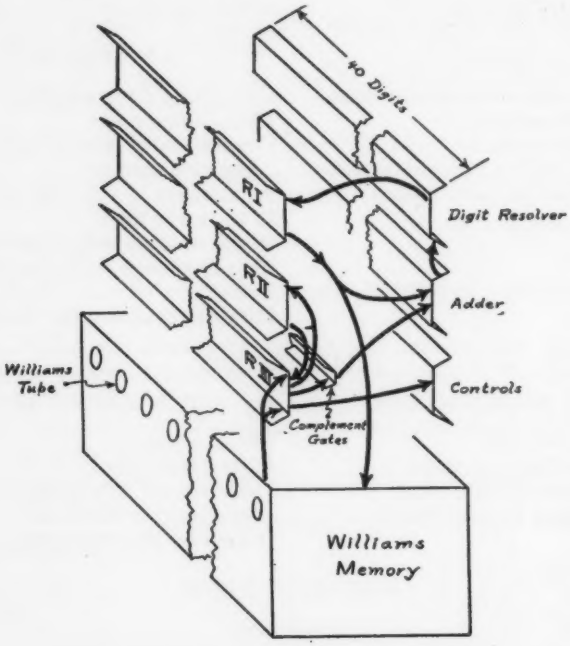
In process

The translation of the preliminary principles of logical design into a working high-speed machine was accomplished by a small group of engineers and technicians directed by JULIAN H. BIGELOW. Bigelow's design disciplines were:

(a) Compactness, achieved by seeking optimum geometrical distribution of components, to minimize the effect of the purely parisitic interconnections.
(b) Minimization of the number of tubes, achieved by (a) along with the elimination of logical redundancies and special purpose circuitry.
(c) Logical operations independent of signal waveforms.

(d) Sequential control operation with each key operation initiated by the safe completion of the preceding operation.
(e) Safe circuit operation between essentially zero duty cycle and its maximum safe repetition rate since the duty cycle over a long period of varied computations could not be predicted.
(f) All of the above bounded by very conservative design criteria for the components.

The major departure of the final machine from the proposals of the 1946 report is the use of the Williams storage phenomena in a cathode ray tube memory. The 40-stage memory with a capacity of 1024 words was developed by JAMES H. POMERENE and Bigelow.



SCHEMATIC OF BASIC CONNECTIONS BETWEEN WILLIAMS MEMORY, CONTROL, AND ARITHMETIC UNITS

FIG. 1.

Structure.

The frontispiece shows a view of the Institute machine.

Figure 1 is a schematic establishing the physical layout of the machine and displaying the interconnections between the Memory, Control, and Arithmetic Units.

The large block at the bottom contains the 40 cathode ray tubes (5CP1A), 20 on each side. Each memory tube has an associated three stage amplifier and an eight tube discriminator unit which controls the beam turn-on.

The upper portion contains the arithmetic and control organs. On the left there are three double rank registers each containing two rows of 40 flip-flops and two rows of gate tubes for entry into them. The left-most stage contains the most significant digit. The machine is designed to accommodate numbers in the range $-1 \leq x < +1$, with negative numbers handled in the complemented representation. The most significant digit thus becomes the sign indicator with the binary point fixed to the right of it.

The register labelled RI is called the Accumulator. It is a double rank shifting register which holds the resident number and then the final sum during an addition, the partial remainders during division and the most significant digits of the partial products during multiplication.

RII is called the Arithmetic Register. It is a double rank shifting register which holds the multiplier and the least significant digits of the partial products during multiplication and the quotient during division.

RIII is a non-shifting Memory Register. In the upper row of 40 flip-flops it receives *numbers* from the memory and presents them to the adder via the 40 complementing gates. It contains the incident number during addition, the multiplicand during multiplication and the divisor during division. In the lower row of flip-flops it receives *order-pairs* from the memory, each half containing a ten digit address and a ten digit instruction. Since order-pairs and numerical data are indistinguishable outside of the process in which they are used, any order-pair may be altered by bringing it from the memory to the arithmetic organ via the upper row of RIII, performing an arithmetic operation on it and storing the result back in the memory.

On the right the row of adder chassis contains 40 parallel circuits each receiving three inputs; the resident digit from RI, the incident digit from RIII (upper) via the complement gates, and a carry digit from the next less significant stage. The adder circuit contains four tubes which propagate a carry to the next stage and reduce the eight possible inputs to four possible voltage levels. The row of chassis above the adder is called the digit resolver and reduces the four voltage levels coming out of the adder to two voltage levels representing a "1" or a "0". This information is then communicated to RI.

The lower row of chassis comprise the control organ containing four major units: the main control, the shift counter, the address counter, and the memory control.

1) The main control establishes the sequence for observing the two halves of any order pair and bringing up a new order. It decodes the order digits and commands the execution of the resulting process. It is completely asynchronous. It does not go through the expense of matrix decoding, but rather utilizes unique logical combinations of several order digits to command the desired process.

2) The shift counter counts the number of shifts in the shifting registers and contains recognition circuits which determine when shifting shall be terminated appropriately to the order being done.

3) The memory control with its associated pulse chain establishes reading and writing routines under the command of the main control, and, independently, routines to control the regeneration of the static information in the memory.

4) The address counter remembers the locations of the last point regenerated on the face of the cathode ray tube and of the last order-pair executed; advancing these addresses by one each time. It is also capable of receiving from RIII, however, the address associated with any instruction. During every memory cycle one of the above addresses is used to command the deflection of the cathode ray beams.

Memory.

During any period of time in which the memory is not specifically involved in delivering information to or receiving information from the arithmetic organ, the memory control reverts back to sequential regeneration of the 32 × 32 words stored on the phosphor surfaces of the 40 cathode ray tubes. The timing of this process is governed by a clock and associated pulse chain, with a 24 μ.s. period. Within this period there is one moment when the memory control is receptive to intervention by the main control. The main control may at this time be specifying one of the following processes:

(a) Bring the next order-pair of a consecutive series of orders into the lower row of 40 toggles in the memory register RIII.
(b) Depart from this consecutive series of orders to bring a new order-pair into the lower row of the memory register RIII from an address specified by a "transfer control" order in RIII.
(c) Bring a number into the upper row of the memory register RIII from the address specified by the order in RIII.
(d) Store the number residing in the accumulator register RI at the address specified by the order in RIII.

The memory control reports back to the main control when it accepts the main control intervention and when it completes any of the actions described in (a)–(d). In this manner the synchronous memory operation is safely incorporated into the asynchronous arithmetic process.

The inherent coupling between adjacent storage locations on the face of any cathode ray tube places a limit on the number of times any point on the raster may be consulted before its neighbors are regenerated. For the Institute machine this number, called the "read around" ratio, is in the neighborhood of 30.

Arithmetic and Control Operations.

The twenty basic orders executed by the Institute machine are listed below with the average time per order measured in a long sequence of similar orders. Many of the processes are operating with inordinate factors of safety. Thus the speed figures below reflect the present arbitrary state of adjustment of the machine, not its maximum capabilities.

(1–8) *Add*.....62 μ.s.

Bring a number from position x in the memory and add it to the number residing in the accumulator with the following choices:

(a) "Clear" or "Hold" where "Clear" signifies the desire to preclear the accumulator to zero.

(b) "Magnitude" or "Number" where "Magnitude" signifies that the absolute magnitude of the quantity at x is to be added.

(c) "−" or "+".

(9) *Multiply:* Multiplier all ones.....990 μ.s.
Multiplier all zeros.....435 μ.s.
Multiplier half ones.....720 μ.s.

Bring a number from position x in the memory and multiply it by the number residing in the Arithmetic Shifting Register, RII. This order has choice of "Round off."

(10) *Divide*.....1100 μ.s.

Divide the number in the Accumulator by a number to be brought up from position x in the memory.

(11) *Load RII*.....65 μ.s.

Bring a number from position x in the memory and place that number in the Arithmetic Shifting Register, RII.

(12) *Left Shift*.....42 μ.s. + 8 μ.s./shift

Shift the numbers in the Accumulator register, RI and the Arithmetic Shifting Register RII, $n(\leq 47)$ places to the left. During this process the contents of RI shift end around into the right end of RII. However, only zeros are propagated through the right end of RI. The number n is stated in the address portion of the order.

(13) *Right Shift*.....42 μ.s. + 8 μ.s./shift

Shift the numbers in the Accumulator register RI and the Arithmetic Shifting Register RII $n(\leq 47)$ places to the right. During this process the contents of RI shift end around into the left end of RII. However, only the sign digit of the original quantity in RI is propagated through the left end of RI. The right shift order has the choice of Round Off. If Round Off is chosen, 2^{-39} is added into the quantity in RI at the end of "n" right shifts and a total of $n + 1$ shifts are executed.

(14) *Store*.....58 μ.s.

Take the number existing in the Accumulator register RI and store it at position x in the memory.

(15) *Parity Unconditional Transfer*.....39 μ.s.

Bring a new order pair into the lower rank of the Memory Register, RIII from position x in the memory and proceed to observe that order which occupies the same half of the order pair as the unconditional transfer being executed.

(16) *Non-Parity Unconditional Transfer*.....39 μ.s.

Bring a new order pair into the lower rank of the Memory Register, RIII from position x in the memory and proceed to observe that order which occupies the half of the order pair *opposite* to that occupied by the unconditional transfer being executed.

(17–18) *Parity and Non-Parity Conditional Transfers*.....39 μ.s.

If the number in the accumulator is positive, execute operations described in 15 and 16.

If the number in the accumulator is negative, ignore the request for transfer and proceed to the next order in sequence.

(19) *RII...> RI*.....50 μ.s.

Take the number in the Arithmetic Shifting Register RII and add it to the quantity in the accumulator register RI. This order includes all of the eight variants described in orders (1–8).

(20) *Input-Output Order*

(a) IBM

(1) Input.....n/100 min.
Transfer 12 n words from n IBM punched cards to the 12 n memory addresses starting at address x.

(2) Output.....n/100 min.
Transfer 12 n words from the memory addresses starting at address x to n IBM cards.

(b) Magnetic Drum

(a) Input.....Approx $(150 + 3n)$ milliseconds.
Transfer 32 n words starting with block n_0 of Track A or B from the magnetic drum to the 32 n memory addresses starting at address x $(1 \leq n \leq 32;\ 0 \leq n_0 \leq 31)$.

(2) Output.....Approx $(150 + 3n)$ milliseconds.
Transfer 32 n words from the memory addresses starting at address x to the n Blocks of track A or B of the magnetic drum following the starting Block n_0.

Operating Experiences.

During the first year of machine operation, a considerable amount of time was spent in strengthening weak points in the machine. Towards the latter part of the year the machine came under control with a routine maintenance program permitting long periods of uninterrupted computation.

The bulk of the machine's useful computation time has been devoted to the solution of sets of partial differential equations representing simplified models of the atmosphere. These problems were formulated by a group led by J. G. CHARNEY. In addition, problems in number theory, matrix inversion, and integral equations have been solved by its circuitry. These problems required about 3×10^8 multiplications with the time split fairly evenly between automatic computation and input-output.

GERALD ESTRIN

Institute for Advanced Study
Princeton, N. J.

Bibliography Z

1016. Bruno de Finetti, *Macchine "che pensano" (e che fanno pensare)*, Pubblicazioni delle Facolta di Scienze e di Ingegneria dell'Universita di Trieste, 1952, 34 p., 23.4 × 29.5 cm.

This brochure contains an expository treatment of the automatic digital computer of our era, seen through the eyes of a mathematician and philosopher. Technical details of machine design, being of minor importance to him, are relegated to an appendix which includes photographs and descriptions of some individual machines and the inevitable summary table of speeds and memory capacities. While this appendix is purposely incomplete and not too stimulating, the main part of the booklet is written with a depth of understanding and a clarity and conciseness of expression seldom found in this field. To a reader who knows nothing of computing machines, it explains the relationships between their development and such fields as neurophysiology and psychology, formal logic, information theory, and cybernetics.

It explains the basic functions and components of automatic digital computers, the essential steps involved in programming and coding, the design of "subroutines," and their assembly into comprehensive routines. It touches upon the major classes of computational problems, evaluation of functions and use of tables, matrix inversion and numerical integration of ordinary and partial differential equations. There is a section on relaxation methods and one on the Monte Carlo method. In each of these subjects the fundamental features are adequately brought out in just a few sentences. While it is not clear why the author arranged his topics the way he did, their selection and presentation leaves nothing to be desired.

F. L. Alt

NBSNAML

1017. F. Gruenberger, *Computing Manual*, The University of Wisconsin Press, Madison, 1952, 123 p., 15.5 × 23.5 cm. Price $2.00.

This manual was used as a text in a course entitled "Theory and Operation of Computing Machines" held at the University of Wisconsin during 1951 and 1952. The author discusses IBM techniques for chi-squared analysis, random sampling, questionnaires, bivariate tables and differencing. He also presents miscellaneous operating hints for the IBM laboratory worker.

R. K. Anderson

NBSCL

1018. H. D. & V. R. Huskey, "Electronic computers aiding management control," *The Journal of Accountancy*, Jan. 1952, p. 69–75.

In this very informative article, the authors point out the many economies which may be effected by the use of electronic computation in business administration. Not only would present-day calculations be carried out more efficiently, but the savings in labor, time, and expense would enable various organizations to undertake the solutions of problems hitherto considered prohibitive, for example, the forecasting of trends, as based upon ac-

cumulated data. The recent election prediction carried out by the UNIVAC amply confirms this statement made several months before Election Day by the authors.

The authors indicate some of the factors which account for the economies realized from the use of the electronic computers, such as the immense speed of internal computation. For example, in the time it takes to say the word "multiplication," some computers can perform about 2500 multiplications, or 6000 additions, of pairs of ten-digit numbers. Moreover, these machines are constructed to recognize, and cause automatically the execution of, instructions to perform numerical operations. The resulting elimination of human intervention is a potent time-saver indeed; it also permits a considerable reduction in personnel and in space requirements.

A brief but illuminating history of high-speed computing machines concludes this excellent article.

The reviewer is left with the feeling, however, that the authors give too optimistic a concept of the prowess of the present-day machines as applied to business needs. The prodigious feats of computation which they have exhibited do not apply to all classes of problems. Where defense tasks are concerned, the astounding reduction in time required for their solution is an advantage far outweighing any other considerations. But business managers cannot expect any spectacular savings in dollars and cents until efficient electronic sorter-collators and files allowing random access have been perfected.

IDA RHODES

NBSCL

1019. H. KENOSIAN, "Unitized pulse circuits speed computer design," *Electronics*, v. 25, no. 10, Oct. 1952, p. 156–157.

These unitized pulse circuits were originally developed in Project Whirlwind at MIT. They have been further developed and made commercially available by the Burroughs Adding Machine Co. The ten types of unitized circuits include generators for half-sine-wave pulses of 0.1 microsecond duration, a flip-flop which can be triggered by the pulses and gives out d-c levels for gating, detectors of coincidence between a pulse and a d-c level or between up to five sources of d-c levels, pulse delays, pulse mixers, and channel selectors. Each unit occupies $3\frac{1}{2}$ inches of height in a 19 inch relay rack. They may be interconnected by 93 ohm coaxial cable into complex systems. For example, the Burroughs Laboratory Computer incorporates nearly 500 of these basic pulse control units.

R. D. ELBOURN

NBSEC

1020. H. KOPPEL, "Digital computer plays NIM," *Electronics*, v. 25, Nov. 1952, p. 155–157.

Four stacks of seven chips each are indicated by lights. A "safe" starting condition makes it possible for the player to beat the machine, but not by simply memorizing plays. No electrical diagram appears, but a block diagram shows adders, binary counters, gates, pulser, and scanner. Other

binary game machines at University of Toronto, NBS Corona Labs, etc., are noted. The win theory of this game is given by BOUTON.[1]

J. HOWARD WRIGHT

NBSEC

[1] C. L. BOUTON, "Nim, a game with a complete mathematical theory," *Annals of Math.*, s. 2, v. 3, p. 35–39.

1021. F. C. MULLANEY, "Design features of the Era 1101 Computer," *Electrical Engineering*, Nov. 1952, p. 1015–1018.

The article gives a brief description of a computer using a magnetic drum memory. The first part deals with the basic facts of the machine: what it consists of, what it does, and how it does it. The second part is devoted to the constructional features of the machine and is provided with illustrations. The third section treats the problem of testing and maintenance. Finally, the operational record for over 7000 hrs. is given and a resume of desirable features of this computer presented.

A. WEINBERGER

NBSEC

1022. OFFICE OF NAVAL RESEARCH, *Digital Computer Newsletter*, v. 4, no. 4, Oct. 1952, 11 pages.

The present status of the following digital computers is treated briefly in this number:

1. Naval Proving Ground Calculators
2. The UNIVAC
3. Whirlwind I
4. Computer Research Corporation Computers
   - CADAC 102-A
   - CRC 105
   - CRC 107
5. Moore School Automatic Computer (MSAC)
6. Raytheon Automatic Computer (RAYDAC)
7. The SWAC
8. Aberdeen Proving Ground Computers
   - The ORDVAC
   - The EDVAC
   - The ENIAC
   - The Bell
   - IBM
9. C.S.I.R.O. Mark I

Under Data Processing and Conversion Equipment:

1. Graph Plotter

1023. L. PACKER & W. J. WRAY, JR., "Germanium photodiodes read computer tapes," *Electronics*, v. 25, Nov. 1952, p. 150–151.

A six-channel device for reading paper tape similar to the teletype device uses germanium photoelectric sensing at a speed of 100 pulses per second, although the electronic circuits are capable of 35,000 per second. Small size and large output of the germanium units are advantageous. Only six

diodes are used, but nonetheless this reviewer suggests that reliability is doubtful on the basis of various reports of instabilities due to the plastic encapsulation of these particular diodes.

J. HOWARD WRIGHT

NBSEC

1024. J. RABINOW, "The notched-disk memory," *Electrical Engineering*, v. 71, 1952, p. 745–749.

One of the increasingly pressing problems of government and business is that of the compact storage of large amounts of information in a quickly accessible form.

This paper describes a storage device consisting of many thin iron oxide coated aluminum disks. Information may be recorded on both sides of the disks as coded magnetic pulses. The disks are arranged in the form of a toroid. When it is desired to read or record information on a particular disk, the magnetic recording heads are automatically swung to the chosen disk and the disk is rotated between the recording heads for one revolution.

It is estimated that such a device consisting of 588 disks 20 inches in diameter would have a capacity of a quarter of a billion binary digits. Test results demonstrate that stored data can be reached in times of the order of half a second.

E. F. AINSWORTH

NBSEC

NEWS

Joint Computer Conference.—On December 10–12, 1952, a second joint annual computer conference and exhibition was held in New York under the auspices of the American Institute of Electrical Engineers, Institute of Radio Engineers, and Association for Computing Machinery. The program for the meeting was as follows:

Wednesday, December 10, 1952	Registration and Exhibits
Morning session	S. B. WILLIAMS, *Chairman*, Burroughs Adding Machine Co., Consultant
Keynote address	N. H. TAYLOR, MIT
Recording techniques for digital coded data	A. W. TYLER, Eastman Kodak Co.
Afternoon session	M. M. ASTRAHAN, IBM, *Chairman*
Converters for teletype tape to IBM cards	G. F. NIELSEN, IBM
Punched card to magnetic tape converter for UNIVAC	E. BLUMENTHAL and F. LOPEY, Eckert-Mauchly Div., Remington Rand, Inc.
Devices for transporting the recording media	R. L. SNYDER, JR., Consulting Engineer
Buffering between input-output and the computer	A. L. LEINER, NBS
Thursday, December 11, 1952	
Morning session	W. S. MACWILLIAMS, *Chairman*, BTL
SEAC input-output system:	
SEAC input-output system	S. GRUNWALD, NBS
Input-output devices used with SEAC	J. L. PIKE
Auxiliary equipment to SEAC input-output	RUTH C. HAUETER, NBS
Operating experience	E. AINSWORTH, NBS

UNIVAC input-output system:	
The Uniservo-tape reader and recorder	H. F. WELSH and H. LUKOFF, Eckert-Mauchly Div., Remington Rand, Inc.
Input devices	L. D. WILSON and E. ROGGENSTEIN, Eckert-Mauchly Div., Remington Rand, Inc.
Output devices	E. MASTERSON and L. D. WILSON, Eckert-Mauchly Div., Remington Rand, Inc.
Afternoon session	J. H. HOWARD, *Chairman*, Burroughs Adding Machine Co.
RAYDAC input-output system:	
RAYDAC input-output systems	W. GRAY and K. RAHLER, Raytheon
System design	L. FERN, Raytheon
Operating experience with RAYDAC	F. DEAN, Raytheon
IBM 701 input-output system:	
Type 701 input-output organization	L. D. STEVENS, IBM
Type 726 magnetic tape reader and recorder	W. S. BUSLIK, IBM
Magnetic tape techniques and performance	H. W. NORDYKE, IBM
Friday, December 12, 1952	
Morning session	J. C. MCPHERSON, *Chairman*, IBM
Survey of analog-to-digital data converters	H. E. BURKE, Consolidated Engineering Corp.
Survey of mechanical type printers	J. HOSKEN, A. D. Little, Inc.
Survey of nonmechanical type printers	Lt. R. J. ROSSHEIM, GWU
Anelex printer	L. ROSEN, Anderson-Nichols Co.
Eastman printer	R. G. THOMPSON, Eastman Kodak Co.
Afternoon session	J. G. BRAINERD, *Chairman*, Moore School of Electrical Engineering, Univ. of Penn.
Ferranti input-output equipment	B. W. POLLARD, Ferranti Ltd., London
Garment tag equipment	O. G. HESSLER, Sears Roebuck and Co.
Numerically controlled machine tool	W. PEASE, MIT
Summary and forecast	S. N. ALEXANDER, NBS
Informal discussion	

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z

1025. LEONILDA ALTMAN, "An analog computer as an aid to shower theory calculations," *Rev. Sci. Instruments*, v. 23, 1952, p. 382.

A. c. resolvers are used to solve certain trigonometrical equations.

F. J. M.

1026. S. BOSWORTH, "A new analog computer," *Electronics*, v. 24, no. 8, Aug. 1951, p. 216–224.

This article describes "an inexpensive desk-size electronic differential analyzer" suitable for differential equations with constant coefficients. The device contains 20 d.c. amplifiers, 23 ten-turn potentiometers, and 8 integrating capacitors. The problem is set up on terminal boards which are plugged into the device.

F. J. M.

1027. F. E. Brooks, Jr. & H. W. Smith, "A computer for correlation functions," *Rev. Sci. Instruments*, v. 23, 1952, p. 121–126.

A continuous computer is described for calculating $\int_0^{t_1}(F(t)G(t+\tau) - \bar{F}G(t+\tau))dt$ for a given value of τ, where $F(t)$ and $G(t)$ are given arbitrary functions of t and $\bar{F}$ is the average value of F on the given interval of integration. The correlation function is computed from a number of such expressions. $F(t)$ and $G(t)$ are recorded on magnetic tapes in the form of pulse length modulation of a rectangular wave with a repetition rate of five cycles per second. The mathematical operation required is performed by a multiplier integrator. The integration is performed by a condenser which is fed by a multiplying unit whose output comes through a gate, which is controlled to be off or on by the pulses for one factor. The input to this gate is the result of demodulating the signal for the other factor and obtaining an amplitude modulated signal. The variable τ is introduced by varying the relative position of the two tapes. It is planned to use this in the study of the fading of radio signals due to various factors and other atmospheric studies. An example involving thermistors is given.

F. J. M.

1028. F. B. Daniels, "A planimetric method of harmonic analysis," *Rev. Sci. Instruments*, v. 23, 1952, p. 369–370.

This article gives a formula in terms of the coefficients of the Fourier series of $f(x)$ for an integral which is readily evaluated by a planimeter. Let $\varphi_k(x) = (-1)^j$ if $j/2k\pi \leq x < (j+1)/2k\pi$. The integral obtained by the planimeter is $\int_0^{2\pi}\varphi_k f dx$. This can be considered as an approximation to a_k, the coefficient of $\cos kx$ in the Fourier series for $f(x)$. A phase shift yields an approximation to b_k, the coefficient of $\sin kx$.

F. J. M.

1029. E. A. Goldberg, "Step multiplier in guided missile computer," *Electronics*, v. 24, no. 8, August 1951, p. 120–124.

A set of relays can be used to obtain a conductance whose value is determined by a binary coded number $.\alpha_1\alpha_2 \cdots \alpha_n$. Thus if $\alpha_j = 1$, the j'th relay contact will introduce a conductance with value 2^{-j} units. Now suppose the state of the j'th relay is determined by the $(n-j)$'th stage of a binary counter. Apply a fixed voltage to this conductance and compare the resulting current with another current whose value corresponds to a mathematical variable x. The difference controls gates across the input of the binary counter, so that if there is a significant discrepancy between these currents, an appropriate gate is opened and counting pulses reach the counter causing it to count up or down until the value of the conductance corresponds to the value of the variable x. The same binary counter can be used to control any number of such relay conductances and thus any number of conductances with the value of a specified mathematical variable x can be produced. If now, voltages corresponding to other variables, $y_1, \cdots, y_r$, are applied to these conductances, currents with values $xy_1, \cdots, xy_r$ may be obtained.

The actual conductances are T networks and 11 stages are used. The counting pulses are at a rate of 1 kilocycle and the relay conductances in-

volve a specially developed relay with a response time of approximately 100 microseconds. The circuit for the binary counter is also discussed.

These multipliers were developed as part of "Project Typhoon" for the U. S. Navy. They are used in the Typhoon guided missile simulator where multipliers are needed with a precision higher than that available from servo multipliers. The article also discusses various features of this computer, which is essentially a large electronic differential analyzer using 445 stabilized d.c. amplifiers. These can be used for either integration or summation. There are 19 computing servo units and 36 multiplications can be made by the above described stepmultipliers (involving the 9 distinct x factors).

F. J. M.

1030. A. E. HASTINGS & J. E. MEADE, "A device for computing correlation functions," *Rev. Sci. Instruments*, v. 23, 1952, p. 347–349.

This device produces the integral $\int_{-T}^{T} f(t)f(t+\tau)dt$. The function $f(t)$ is recorded on a magnetic tape by amplitude modulation and read by two heads whose separation corresponds to τ. One head signal is used to produce a current, the other a voltage and the result is obtained from a wattmeter.

F. J. M.

1031. INSTITUTE OF GAS TECHNOLOGY, "Gas distribution system calculator," *Rev. Sci. Instruments*, v. 23, 1952, p. 61.

A photograph of a special purpose network analog computer and a short description are given.

F. J. M.

1032. R. D. MINDLIN & M. G. SALVADORI, "Analogies," *Handbook of Experimental Stress Analysis*. John Wiley & Son, New York, 1950, Chapter 16, p. 700–827.

When the behavior of two physical systems can be described in identical mathematical form, they are said to be analogous. If experiments and measurements can be made with greater ease and accuracy on one of the systems, then it may be used as a computing device for the solution of complex problems arising in the more intractable system. In this article, the authors present a comprehensive and detailed survey of those analogies between elasticity and other physical phenomena which have been used in the quantitative determination of states of stress in elastic bodies. Each analogy is treated carefully, deriving first the mathematical equations governing both the original physical system and the analogous system. The corresponding quantities in the two systems and the required scale factors are then readily recognized. Methods of physically realizing and measuring quantities in the analogous system are then described, including a discussion of the overall accuracy obtainable. An excellent bibliography containing sixty-seven entries is appended.

The earliest use of analogies in elasticity theory was apparently in the determination of stresses in a cylindrical bar bent by a terminal load or twisted by terminal couples. At a sufficient distance from the end of the bar, the problem may be reduced to the solution of Laplace's or Poisson's equation with suitable boundary conditions. Therefore, any of the other

physical phenomena satisfying these equations may be used as analogies. PRANDTL, in 1903, suggested the use of a homogeneous membrane, under uniform tension per unit of length and subjected to a normal pressure per unit of surface area, the deflection of which satisfies Poisson's equation, provided the slope of the membrane relative to a plane is everywhere small. This analogy has been extensively employed and a full description is given of methods of its physical realization, of fixing its boundaries so that the boundary conditions are analogous to those of the original problem and methods of measuring deflections of the membrane accurately. Soap films have been used primarily, although the use of rubber and the separation surface of two immiscible liquids are described. The steady-state electrical potential in a thin plate of constant thickness, which satisfies Laplace's equation, has also been utilized.

The use in analogies of continuous physical systems such as the membrane or thin conducting plate, as opposed to lumped parameter systems, has the advantage that the analogous system satisfies the same system of equations as the original system, whereas the lumped parameter system will satisfy only a finite difference approximation to those equations. However, the use of continuous systems requires usually the construction of new apparatus (molds, etc.) for the solution of each problem and delicate experimental technique for accurate measurements. For these reasons, it appears to the reviewer, that a lumped D. C. network would represent a more flexible analogy to the torsion problem than the membranes or conducting plates heretofore used.

Analogies for the solution of many other problems in elasticity are discussed. Among them are electrical analogies for the torsion of bars of varying circular section and analogies between problems in elasticity which are difficult to study experimentally and other elasticity problems which are more susceptible to direct measurement. Thus an analogy exists between the stresses produced by a two dimensional steady-state temperature distribution and the stresses produced by dislocations, and the latter problem is easier to treat experimentally.

Considerable space is devoted to a lucid exposition of the A. C. electric network analogue devised by G. KRON for the general case of static plane stress. A lumped parameter system, utilizing inductances and capacitances, it satisfies a finite difference approximation to the equations governing the continuous system. Its great generality is very attractive. However, the analogy requires that the inductances and capacitances have no resistive losses. In an actual realization of the analogue described in this article, resistive losses resulted in large errors.

J. H. WEINER

Columbia University
New York 27, New York

1033. REEVES INSTRUMENT CORPORATION. *Project Cyclone Symposium II on Simulation and Computing Techniques.* New York City, April 28–May 2, 1952, 237 p.

The conference reviewed below was the second symposium on REAC techniques sponsored by the Reeves Instrument Corp. The first symposium was held in 1951 [*MTAC*, v. 5, p. 250–253]. Project Cyclone at Reeves is

sponsored by the U. S. Navy Bureau of Aeronautics and the U. S. Navy Special Devices Center. The functions of Project Cyclone are: "(1) the development and test operation of a guided missile simulator . . . ; (2) the operation of a computer laboratory where classified problems in other fields may be studied and analyzed; (3) the investigation of new applications for Cyclone equipment; (4) theoretical research to ascertain and improve the reliability and accuracy of computer solutions; and (5) the coordination of the experiences of individual computer groups . . . ."

The 24 papers presented in this publication can be roughly classified into four types. (a) Papers dealing with circuitry, techniques and auxiliary equipment to improve the performance of REAC equipment; (b) Papers dealing with the application of REAC equipment to various physical problems encountered by the various users of the equipment; (c) Papers dealing with computers other than REAC; (d) Papers dealing with basic theory and new equipment.

(a) Circuitry

Paper No. 10, "Some REAC techniques employed at the David Taylor Model Basin" by L. Pode discusses techniques for generating input functions and the use of the REAC as an harmonic analyzer in problems that arose at the DTMB. Paper No. 11, "Simulation studies of a relay servomechanism" by N. P. Tomlinson considers methods for the analog study of a non-linear relay (that is, sampling) servomechanism. Paper No. 13, "Precision in high-speed electronic differential analyzers" by H. Bell, Jr. & V. C. Rideout is concerned with solving linear differential equations with variable coefficients and non-linear equations by analog computers and a comparison of the accuracy of solutions compared with digital solution. Paper No. 15, "Solution of linear differential equations with time-varying coefficients by the electronic differential analyzer" by C. E. Howe, R. M. Howe & L. L. Rauch discusses techniques for solving linear differential equations on the REAC. Examples include Bessel's equation, Legendre's equation and the deflection equation $[EI(x)y'']'' - \mu(x)\lambda^2 y = 0$ of simple beam theory. Paper No. 19 is entitled "Automatic REAC operation for statistical studies" by R. R. Bennett & A. S. Fulton. Determination of the response of a non-linear system to statistical inputs can most conveniently be carried out experimentally, that is, on an analog computer. Methods for changing a Gaussian distribution by passing the noise through a function generator are discussed. Paper No. 21, "Checking analogue computer solutions" by W. F. Richmond, Jr. & B. D. Loveman considers procedures to verify analog computer solutions of linear and non-linear differential equations. Linear equations are treated by frequency (steady state response) or selected root method (transient response). For non-linear systems "dynamic substitution" is used. It consists in breaking-up the time interval into small pieces and substituting the solution on each small sub-interval into the original differential equation. Paper No. 24, "Modifications and additions to the REAC" by J. W. Follin, Jr., G. F. Emch & F. M. Walters discusses circuitry details and auxiliary equipment for improving the operation of the REAC. These include a low frequency noise generator, a programmer for turning the REAC and recorder on and off, air conditioning the racks,

power supply changes, circuit changes in amplifiers and techniques of interconnection of machines. Paper No. 25, "REAC servo response" by A. H. MILLER discusses methods for introducing variable coefficients and non-linearities into REAC equipment by the use of servos. A study of the dynamic characteristics of servos for use in the above connection is made. Paper No. 26 is "Applications of differential relays to solution of REAC problems" by L. M. WARSHAWSKY & W. BRAUN. A differential relay is a single-pole double-throw relay at the output of an operational amplifier. Its uses include providing a time-controlled series of switching operations and a simulation of "dead" zone where the operating limits are variable. Paper No. 27, "Techniques applied to the design and test of linear filters" by G. NESTOR is concerned with problems met in building and testing a Wiener filter. Paper No. 28, "The role of diodes in an electronic differential analyzer" by C. D. MORRILL & R. V. BAUM discusses the use of diodes to simulate certain types of non-linearities.

(*b*) *Applications*

Paper No. 7: "REAC solution of problems in structural dynamics" by C. W. BRENNER. In aircraft design one can no longer consider the air-frame as a rigid body. Many components designed only for maximum static loads fail in service due to transient loading. This paper considers a simplified associated problem of a simply-supported beam struck by a pressure wave (a) Normal to the beam, (b) Parallel to the beam. For case (a) there arises the differential equation $\phi_r'' + k^2 r^4 \phi_r = \epsilon^{-t/t_1}(1 - t/t_1), r = 1, 2, \cdots$, where ϕ_r is the normalized displacement of the rth mode of vibration and k is a parameter. A similar equation holds for case (b). These equations are solved on the REAC. A discussion of the circuitry details and the troubles that were encountered together with their solutions, is presented.

Paper No. 8: "The use of an analog computer and feedback theory for the solution of structural problems in the static case" by G. J. MARTIN & L. M. LEGATSKI. Structural frameworks (pinned frames, statically indeterminate structures) give rise to certain systems of linear algebraic equations of the form $\sum_{j=1}^{n} a_{ij}\theta_j = M_i$, $i = 1, \cdots, m$, where the θ's are angles representing the rotations of various members of the structures and the M's are moments. These equations bear great similarity to usual mesh equations in electrical network theory and are solved on the REAC using a network analog.

Paper No. 9: "Application of the electronic differential analyzer to eigenvalue problems" by G. M. CORCOS, R. M. HOWE, L. L. RAUCH & J. R. SELLARS. Determination of eigenvalues for Sturm-Liouville systems, fourth order equations and partial differential equations. The partial differential equations are reduced to ordinary differential equations by separation of variables. Since, in general, there are two-point boundary conditions, the method is the familiar technique of satisfying the initial conditions at one point and varying the solution until the boundary conditions at the second point are satisfied. The authors' experience leads them to conclude that one or two end conditions are easy to handle, three end conditions give considerable trouble and four or more end conditions are apt to be hopeless. Four examples are considered.

Paper No. 12: "Use of the REAC as a curve fitting device" by C. H. MURPHY. In certain problems in aerodynamics it is necessary to determine the coefficients of a differential equation from discrete points on its solution curve. The use of the REAC in this connection is described, based on certain linearized differential equations of yawing motion.

Paper No. 14: "An application of analogue computers to problems of statistical analysis" by H. H. LANING, JR. and R. H. BATTIN. The response $x(t)$ of a linear system to a forcing function $f(t)$ is given by $x(t) = \int_{-\infty}^{t} W(t, \tau) f(\tau) d\tau$ where $W(t, \tau)$ is the impulsive response (Green's function, weighting function) of the system. A method for computing this quantity is discussed using the concept of the adjoint system.

Paper No. 23: "Solution of partial integral-differential equations of electron dynamics using analogue computers with storage devices" by C. C. WANG. In certain problems of electron dynamics there arise simultaneous partial integral differential equations involving two independent variables. How to adapt such problems to REAC analog computers is discussed.

(*c*) *Non-REAC computers*

Paper No. 16: "JAINCOMP computers and their application to simulation problems" by D. H. JACOBS. Description of the uses of the JAcobs-INstrumentCOMPany digital computer. It is a compact (17 × 21 × 30 inches), light (110 lbs.) electronic digital computer. The operations of the machine are very rapid (adds two 40 binary digits in eight microsec). Paper No. 17: "The decimal digital differential analyzer CRC-105 as a tool for simulation and checking analogue computer solutions" by E. WEISS. Description of the DDDA and its advantages and disadvantages as compared with electronic analogue computers and mechanical computers. It operates on a decimal number system, has 60 integrators and a maximum accuracy of six digits plus sign in each integrator. There are twelve input and twelve output channels.

Paper No. 18: "Problems encountered in the operation of the M. I. T. flight simulator" by W. W. SEIFERT & H. JACOBS, JR. The title accurately describes the paper. The simulator is located at the DACL (Dynamic Analysis and Control Laboratory) at M. I. T. Paper No. 22: "Analogue computation of blade designs" by D. B. BREEDON, M. M. MATTHEWS & E. L. HARDER. Calculation of natural frequencies and stresses in vibrating beams by the Westinghouse analog computer (Anacom) and checks made on a digital machine (IBM Calculating Punch, Type 602-A).

(*d*) *Theory*

Paper No. 20: "Mathematical error analysis for continuous computers" by F. J. MURRAY. This paper presents certain aspects of the work done by the author and the reviewer on the mathematical foundations of an error theory applicable to mathematical machines. The complete results will appear elsewhere.

Paper No. 29: "A high accuracy time division multiplier" by E. A. GOLDBERG. If a train of rectangular pulses is generated, then the DC component is proportional to the amplitude of the pulses multiplied by the duty cycle. This principle is not new. However, by ingenious circuitry including

the use of a feed-back system for establishing accurate timing and using a current switch essentially independent of tube characteristics, an accurate multiplier is obtained which should be useful in electronic analog equipment. Using precision resistors, stable voltage sources and steep switching pulses an accuracy of about $\pm$ 0.01% of full scale (75 volts) for a maximum prf of 2000 cps is obtainable.

Paper No. 30: "An AM-FM electronic analog multiplier" by W. A. McCool. A signal having both amplitude modulation and frequency modulation is applied to an (AM) detector and a (FM) discriminator. If it is desired to multiply the two quantities X and Y, then the output of the discriminator can be made to be $X = Y_0\Delta f$, and if the output of the detector is forced to match the X input plus the reference voltage Y_0, then its output is $Z' = (Y_0 + Y)\Delta f$ and $Z = Z' - X = Y_0^{-1}XY$. (The frequency deviation Δf is the same for both detectors.) This principle has been used before. By the use of feed-back high gain amplifiers, tube characteristics have only a secondary effect. A mathematical analysis is made of the circuit showing the dependence of Z on the circuit parameters. An experimental model is in the process of being developed at the Naval Research Laboratory.

K. S. Miller

New York University
New York, New York

1034. Denman Shaw & R. N. Weltmann, "A logarithmic counting-rate recorder," *Rev. Sci. Instruments*, v. 23, 1952, p. 528–531.

This device uses a spiral cut on a cylinder to record the logarithm of the time for a specified number of counts from a Geiger counter and scaler.

F. J. M.

1035. S. Softky & J. Jungerman, "Electrolytic tank measurements in three dimensions," *Rev. Sci. Instruments*, v. 23, 1952, p. 306–307.

The authors indicate the techniques necessary to obtain good results in the three dimensional case and obtain in a specific example, an accuracy of .2% in the potential and 2% in the field measured.

F. J. M.

1036. A. W. Vance & C. C. Shumard," Superregulated power supplies," *Electronics*, v. 24, no. 12, Dec. 1951, p. 109–113.

This article describes the power supplies used in the R.C.A. Project Typhoon guided missile simulator, built for the U. S. Navy.

F. J. M.

1037. G. L. Weissler, A. W. Einarsson & J. D. McClelland, "A semi-automatic graphical computer with use in spectroscopy," *Rev. Sci. Instruments*, v. 23, 1952, p. 209–212.

The computer is a simple but ingenious device for graphing a prescribed function φ of a function $f(x)$ where $f(x)$ is given by a graph. The graph of $y = f(x)$ appears on a sheet of paper which passes over a line parallel to the y axis. Above this line is a transparent plate which moves parallel to the x axis. On this plate, one has a graph $x = \varphi(y)$ and there is a recorder moving

synchronously with the f graph which records the x motion of the plate. The operator moves the plate so that the $y = f(x)$ and $x = \varphi(y)$ graphs have the same ordinate y on the above-mentioned line and hence the desired output $\varphi(f(x))$ is obtained from the recorder. The article contains details of the construction and of the application mentioned in the title.

F. J. M.

NOTES

147.—Stability of Difference Relations in the Solution of Ordinary Differential Equations. In a recent communication, J. Todd[1] demonstrated the danger of replacing a differential equation, for computational purposes, by a difference equation of higher order. H. Rutishauser[2] has since given some general criteria for determining the stability of difference approximations to ordinary differential equations. In the present note, some standard step-by-step methods of integrating ordinary linear differential equations are examined for stability.

Let a linear differential equation be replaced by a finite difference approximation of order p (i.e., one involving $p + 1$ tabular values). Then the nth tabular entry is calculated from

$$y_n + A_1 y_{n-1} + A_2 y_{n-2} + \cdots + A_p y_{n-p} = 0, \tag{1}$$

where $A_1, A_2, \cdots, A_p$ are functions of x and of the interval length h. Now suppose the errors existing in the entries $y_{n-p}, y_{n-p+1}, \cdots, y_{n-1}$ are $\epsilon_{n-p}, \epsilon_{n-p+1}, \cdots, \epsilon_{n-1}$ respectively, then the consequent error in y_n is ϵ_n where

$$\epsilon_n + A_1 \epsilon_{n-1} + A_2 \epsilon_{n-2} + \cdots + A_p \epsilon_{n-p} = 0. \tag{2}$$

Consider also for convenience that the above errors result entirely from errors in the initial values $y_1, y_2, \cdots, y_p$. Then the general error given by equation (2) is

$$\epsilon_n = a_1 \lambda_1^n + a_2 \lambda_2^n + \cdots + a_p \lambda_p^n, \qquad (n > p),$$

where $a_1, a_2, \cdots, a_n$ are constants and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the roots of the auxiliary equation

$$\lambda^p + A_1 \lambda^{p-1} + A_2 \lambda^{p-2} + \cdots + A_p = 0. \tag{3}$$

The condition for stability is that all the roots of equation (3) lie inside or on the unit circle.

Todd[1] considered the differential equation

$$y'' = -y,$$

and its fourth order central difference replacement

$$y_n - 16y_{n-1} + (30 - 12h^2)y_{n-2} - 16y_{n-3} + y_{n-4} = 0.$$

As h approaches zero, the roots of the corresponding auxiliary equation tend to 1, 1, $7 - \sqrt{48}$, and $7 + \sqrt{48}$, the last root quoted being responsible for the instability found by Todd. The fourth order backward difference

formula, however, is

$$(35 + 12h^2)y_n - 104y_{n-1} + 114y_{n-2} - 56y_{n-3} + 11y_{n-4} = 0,$$

with auxiliary equation roots tending to 1, 1, $\sqrt{11/35}$, and $\sqrt{11/35}$ as h approaches zero. This formula is thus stable at least for sufficiently small h. In general, because of the smaller coefficients employed in calculating y_n from a backward difference formula, the chance of multiplying an error will be correspondingly less. Thus in any step-by-step method, stability is more likely to result from using backward than central differences.

The use of backward difference formulae, however, does not ensure stability, for consider the equation

$$y' = -y, \tag{4}$$

where y is to be computed for increasing x. Write

$$hy_0' = \nabla y_0 + \nabla^2 y_0/2 + \nabla^3 y_0/3 + \cdots + \nabla^n y_0/n, \tag{5}$$

leading to the auxiliary equation,

$$h + (1 - 1/\lambda) + (1 - 1/\lambda)^2/2 + (1 - 1/\lambda)^3/3 + \cdots + (1 - 1/\lambda)^n/n = 0. \tag{6}$$

It can be shown that for $h = 0$ the roots of equation (6) other than $\lambda = 1$ have modulus less than unity for $n \leq 6$. For $n = 7$, there is a pair of conjugate complex roots approximately equal to $\pm i$. For $n \geqq 8$ there will be at least one pair of conjugate roots of modulus greater than unity. (The greatest pair is found easily by GRAEFFE's root-squaring method.) Table I demonstrates the instability of the twelfth order backward difference formula applied to equation (4). The values of e^{-x} at decimal intervals from 0 to 1.1 required to start the computations, were taken from five figure tables. The theoretical and computed values from 1.2 to 2 are given in rows (1) and (2) respectively in table I.

Table I

x	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
(1)	.30199	.27253	.24660	.22313	.20190	.18268	.16530	.14957	.13534
(2)	.30148	.27341	.24708	.22244	.20337	.18665	.16017	.14168	.16065

Next consider the stability of the integration formulae due to ADAMS and MOULTON. Adams' method is based on the formula

$$(y_1 - y_0)/h = y_0' + 1/2\nabla y_0' + 5/12\nabla^2 y_0' + 3/8\nabla^3 y_0' + 251/720\nabla^4 y_0' + \cdots \tag{7}$$

where a sufficient number of starting values for y, y' is supposed computed by an independent method (e.g., by Taylor series). Consider again the first order equation (4). Adams' formula leads to the auxiliary equation

$$F(\lambda) = \lambda - 1 + h\{1 + 1/2(1 - 1/\lambda) + 5/12(1 - 1/\lambda)^2 + 3/8(1 - 1/\lambda)^3 + 251/720(1 - 1/\lambda)^4\} = 0, \tag{8}$$

where only fourth differences are retained and it is clear that there is stability as h tends to zero. Now from equation (8), $F(-\infty) < 0$, and $F(-1) = -2 + 551h/45$. There is therefore a root of modulus greater than unity when h exceeds 90/551, and the method is stable only for sufficiently small tabular interval. Moreover if higher order differences are retained, the maximum value of h for which the method is stable is decreased.

Similar arguments show that Moulton's method based on the formula

$$(9)\quad (y_0 - y_{-1})/h = y_0' - 1/2\nabla y_0' - 1/12\nabla^2 y_0' - 1/24\nabla^3 y_0' - 19/720\nabla^4 y_0' - \cdots$$

is also unstable for large values of the tabular interval when differences higher than the first are retained. The upper limit on h for stability for a given number of differences is very much higher than in Adams' method.

It has been remarked by RUTISHAUSER[2] that the error equation, corresponding to a non-linear differential equation of the form

$$y^{(n)} = f(x, y, y^{(1)}, \cdots, y^{(n-1)}),$$

is linear. The above arguments with certain modifications can therefore be applied to the stability associated with equations of this form.

A. R. MITCHELL
J. W. CRAGGS

Department of Mathematics
St. Andrews University
Scotland

[1] J. TODD, "Solution of differential equations by recurrence relations," *MTAC*, v. 4, 1950, p. 39–44.
[2] H. RUTISHAUSER, "Uber die Instabilität von Methoden zur Integration gewöhnlicher Differentialgleichungen," *Zeit. angew. Math. Phys.*, v. 3, 1952, p. 65–74.

148.—TWO NON-ELEMENTARY DEFINITE INTEGRALS. The two integrals in question are

$$F(x) = \int_0^x t^t dt, \qquad G(x) = \int_0^x t^{-t} dt$$

and are of interest because of the peculiar branching properties of the integrands and because they lead to series with unusually rapid convergence. Integrals of these types have been encountered in some recent studies of transients in networks. They can be evaluated numerically as follows.

As usual, we interpret t^t as

$$e^{t \log t} = \sum_{n=0}^{\infty} (t \log t)^n / n!$$

The integral of the general term of this series

$$I_n = \frac{1}{n!} \int_0^x (t \log t)^n dt$$

can be expressed in terms of the complete and incomplete Gamma function by means of the transformation

$$u = -(n+1) \log t.$$

In fact

$$(-1)^n(n+1)^{n+1}I_n = 1 - \frac{\Gamma_y(n+1)}{\Gamma(n+1)}$$

where

$$\Gamma_y(n+1) = \int_0^y e^{-u}u^n du$$

and

$$y = -(n+1)\log x.$$

However, since available tables[1] of the incomplete Gamma function are not convenient for this application, it was decided to use a direct evaluation of the series.

The integral for I_n can be expressed as a sum of $n+1$ terms as follows:[2]

$$I_n = x^{n+1}\sum_{r=0}^{n}\frac{(-1)^r(\log x)^{n-r}}{(n+1)^{r+1}(n-r)!}.$$

Summing over n and collecting the coefficient of each power of $\log x$ we obtain the double series

$$F(x) = \sum_{n=0}^{\infty} I_n = \sum_{r=0}^{\infty}(\log x)^r/r!\sum_{s=1}^{\infty}(-1)^{s+1}x^{s+r}(s+r)^{-s}.$$

Similarly

$$G(x) = \sum_{n=0}^{\infty}(-1)^n I_n = \sum_{r=0}^{\infty}(-1)^r(\log x)^r/r!\sum_{s=1}^{\infty}x^{s+r}(s+r)^{-s}.$$

When $x = 1$ we have the unusually rapidly converging series

$$F(1) = 1 - 2^{-2} + 3^{-3} - 4^{-4} + \cdots,$$
$$G(1) = 1 + 2^{-2} + 3^{-3} + 4^{-4} + \cdots.$$

The following values of $F(x)$ and $G(x)$ were computed to 10D by the above double series and checked by numerical integration using tables of fractional powers.[3] Values have been rounded off to 8D. Most of the calculation was made by Mrs. JOAN M. CLAY.

x	$F(x)$	$G(x)$
0	0.0000 0000	0.0000 0000
.1	0.0870 9546	0.1152 6443
.2	0.1625 6731	0.2478 5617
.3	0.2334 0100	0.3890 5018
.4	0.3027 3442	0.5332 8125
.5	0.3726 1486	0.6763 8768
.6	0.4446 5226	0.8152 2316
.7	0.5202 8866	0.9474 7031
.8	0.6009 4326	1.0715 0820
.9	0.6881 0902	1.1862 9918
1.0	0.7834 3051	1.2912 8600

M. S. CORRINGTON

Radio Corporation of America
Camden, N. J.

[1] KARL PEARSON, *Tables of the Incomplete Γ-Function*. London, H. M. Stationery Office. (Reissue 1934; London, Biometrika Office, University College.)
[2] W. GRÖBNER & N. HOFREITER, *Integraltafel*. Erster Teil, Unbestimmte Integrale. Vienna, 1949, p. 111, formula 2a.
[3] NBSMTP, *Tables of Fractional Powers*. New York, Columbia University Press, 1946.

149.—The CANON DOCTRINAE TRIANGVLORVM (1551) of RHETICUS (1514–1576). Some facts with reference to this excessively rare publication have been given in material about PITISCUS and RHETICUS in *MTAC*, v. 3, p. 394, 396, 553–554, 559–560. It is here noted that the only copies known to have been preserved were in the Bibliothèque Nationale and British Museum. DEMORGAN had a copy in 1845 when he published[1] a description of the work, but this was doubtless in his Library at the University of London, destroyed during the recent World War.

In Catalogue 19, 1952, of the London bookseller E. WEIL, a copy was offered for 27 £ 10 s. Mr. WILLIAM D. MORGAN, of 1764 St. Anthony Ave., St. Paul 4, Minnesota, was so fortunate as to secure this item for adding to his already valuable collection (see *MTAC*, v. 3, p. 562–563). Since Mr. Morgan graciously loaned this precious work to me that a microfilm copy might be made for the Brown University Library, I take the opportunity to add a little to the information already published in *MTAC*. The complete title is as follows: *Canon Doctrinae Triangvlorvm. Nvnc primvm a Georgio Ioachimo Rhetico, in lvcem editvs, cvm privilegio imperiali, Ne qvis haec intra decennivm, quacunque; forma ac compositione, edere, neve sibi vendicere aut operibvs suis inserere ausit. Lipsiae ex officina Wolphgangi Gvnteri. Anno M.D. LI.* In the title page is an obelisk with a man drawing a diagram on the base.

The back of the title page is blank; then follows a page of Latin verses. On the back of this page is the first of 14 pages of 7D tables of the six trigonometric functions, at interval 10′, arranged for the first time in semi-quadrantal display. The degrees are in black, and the minutes and differences are in red. This is the first table in which all trigonometric functions are brought together. Rheticus was the first to define trigonometric functions by means of a right-angled triangle without any reference to a circle.

Immediately following the tables are 6 pages of dialogue between Philomathes, a supposed friend of Rheticus, and Hospes, his pupil. The pupil asks what the intention of the book is, and is answered at length. He suggests that, perhaps, the intention may be to complete the system of Copernicus, by publishing tables from it resembling those then in use. But he is answered that Rheticus would rather that Copernicus himself had not done so much in this line, as he thereby diminished the geometrical practice of the learner, and so on.

An undated 1580 reprint of the *Canon* is in the British Museum.

The copy of the *Canon* before me has evidently had its pages trimmed; but the present size of its pages is 15.8 × 22.5 cm.

R. C. ARCHIBALD

Brown University
Providence, R. I.

[1] DEMORGAN, "On the almost total disappearance of the earliest trigonometrical canon," RAS, *Mo. Not.*, v. 6, 1845, p. 221–228; reprinted with an addition in *Phil. Mag.*, s. 3, v. 26, 1845, p. 517–526.

150.—Square Root on the 602A. The square root setup in the IBM 602A manual [8th ed., p. 81–84] extracts six digit roots of eight digit base numbers at the rate of approximately two roots per minute, using the Newton-Raphson scheme with a fixed (six) number of iterations, with starting values of 3, 30, 300, or 3000, depending on the size of the base number. The 602A must be equipped with division circuits.

About 75 percent of the 602A's in use do not have division circuits. Square roots can be calculated by many iterative schemes, however, of which one of the simplest to program is

$$x_{i+1} = x_i + \tfrac{1}{2}(N - x_i^2) \tag{1}$$

where N is the number whose square root is required, and is taken such that $0 \leq N \leq 1$. The starting value, x_0, is taken as N. The iterations are terminated when

$$|x_{i+1} - x_i| < \epsilon \tag{2}$$

where ϵ is a predetermined small number (say 5 in the 4th decimal place). The error is then less than or equal to $\epsilon N^{-\frac{1}{2}}$.

The rate of convergence is given by the following inequality where $N^{\frac{1}{2}} - x_k$ is the error after the kth iteration:

$$|N^{\frac{1}{2}} - x_k| < (N^{\frac{1}{2}} - N)(1 + N - N^{\frac{1}{2}})^k.$$

One notices that convergence is slow for small values of N.

For N, a 6 digit number, and $\epsilon = 10^{-4}$, square roots to 4D were obtained in the following times.

N	time in seconds
0.9	25
0.8	29
0.7	38
0.6	50
0.5	56
0.4	63
0.3	75
0.2	103
0.1	157

Starting values, x_0, could, of course, be gang punched from a master deck, which would make the setup considerably faster. However, the order of the deck is then disturbed. If the order of the deck is not critical, the remarks below are pertinent; moreover, the entire job (for three digit roots) can be done readily from a master deck of 1000 cards, by collating and gang punching.

If division is available and it is possible to order the cards on the numbers N, then the formula

$$x_{i+1} = \tfrac{1}{2}(x_i + N/x_i) \tag{3}$$

gains speed if the starting value for each card is the result from the previous

card. Approximately eight cards per minute can be calculated, with eight digit N's and square roots to 4D.

The answer from the previous card is taken as x_0. The chief problem in the wiring is to obtain a starting value for the first card, since the 602A performs all the programming wired, in dummy form, before the first card is read. If the cards are in descending order on N, the first starting value can be taken as N, provided that the dummy programming can be skipped. This latter can be accomplished by wiring to "read" from an early program through the normal side of a selector which is then latched for the remainder of the run.

Diagrams of the setups used at the Numerical Analysis Laboratory will be published shortly.

J. HOLLINGSWORTH
F. GRUENBERGER

Numerical Analysis Laboratory
University of Wisconsin
Madison, Wisconsin

151. A NUMERICAL STUDY OF A CONJECTURE OF KUMMER.—The generalization to the cubic case of the well-known (quadratic) Gauss sum was first investigated by KUMMER.[1] He showed that the expression

$$x_p = 1 + 2\sum_{\nu=1}^{(p-1)/2} \cos(2\pi\nu^3/p) \tag{1}$$

for all $p \equiv 1 \pmod 3$ satisfies the cubic equation

$$f(x) = x^3 - 3px - pA = 0 \tag{2}$$

where A is uniquely determined by the requirements

$$4p = A^2 + 27B^2, \qquad A \equiv 1 \pmod 3. \tag{3}$$

Equation (2) clearly has three real roots for each p. Kummer classified the primes $p \equiv 1 \pmod 3$ according to whether the Kummer sum is the largest, middle or smallest root of equation (2). He conjectured that the asymptotic frequencies for these classes of p are (in the order above) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$. To check this surmise he calculated the first 45 of the x_p and found the densities to be .5333, .3111, .1556.

This problem was brought to the attention of the authors by E. ARTIN who suggested the desirability of further testing the conjecture since its truth would have important consequences in algebraic number theory.

Accordingly the primes $p \equiv 1 \pmod 3$ from 7 through 9,973 were tested. We give below a summary of the resulting densities. In this tabulation we have arbitrarily divided the primes into six groups of 100 each, designated by I, $\cdots$, VI and a final group of eleven primes, VII.

Number of primes $p \equiv 1 \pmod 3$ such that

Group	x_p is the largest root	x_p is the middle root	x_p is the smallest root
I	54	28	18
II	41	38	21
III	46	33	21
IV	39	32	29
V	43	29	28
VI	44	38	18
VII	5	3	3
Total	272	201	138
Density	.4452	.3290	.2258

These results would seem to indicate a significant departure from the conjectured densities and a trend toward randomness.

The method of calculation was this: Each root of (2) lies in one of the intervals $(-2p^{\frac{1}{2}}, -p^{\frac{1}{2}})$, $(-p^{\frac{1}{2}}, +p^{\frac{1}{2}})$, $(p^{\frac{1}{2}}, 2p^{\frac{1}{2}})$ as may be seen directly from the form of (2) with the help of (3). For each relevant p the expression (1) for x_p was evaluated, its sign was determined and its square compared to p. This determined in which of the three intervals just described the x_p lies. To check that x_p was indeed a solution of (2), (3) and to determine the precision of the evaluation the expression

$$f(x_p)/f'(x_p) \tag{4}$$

was then calculated. This latter check was performed by first transforming the x_p into decimal form for tabulation and then retransforming these results back into binary form before evaluating the expression (4). In this manner both the calculation proper and the conversion to decimal form of the results were checked.

The trigonometric expressions appearing in (1) were evaluated by power series. Each angle was reduced mod 2π and then mod π until it lay between $-\pi/2$ and $+\pi/2$. Then the cosine of $\frac{1}{4}$ of this angle was calculated keeping five terms in the series expansion. The "double-angle" formula for cosines was then used twice to obtain the desired cosine.

The calculation involved about 15 million multiplications counting the checking mentioned above. The values of p were introduced in blocks of 200. The entire calculation was carried out twice to ensure reliability. The authors are indebted to Mrs. ATLE SELBERG who programmed and coded the calculation.

J. VON NEUMANN
H. H. GOLDSTINE

Institute for Advanced Study
Princeton, New Jersey

[1] E. E. Kummer, "De residuis cubicis disquisitiones nonnullae analyticae," *Jn. f. d. reine u. angew. Math.*, v. 32, 1846, p. 341–365.

CORRIGENDA

v. 6, p. 262, *insert* Emch, G. F. 247.
v. 6, p. 265, under Myers *insert* 54.
v. 6, p. 268, under Yowell *insert* 254.
v. 7, p. 31, l. 11 of MTE 218; *for* $-\log p$ *read* $\log p$.

t

con-

f the
rectly
ession
pared
he x_p
e the

rming
esults
anner
f the

power
tween
eping
osines

g the
f 200.
The
coded

IN
NE

a. f. d.

CLASSIFICATION OF TABLES

A. Arithmetical Tables. Mathematical Constants

B. Powers

C. Logarithms

D. Circular Functions

E. Hyperbolic and Exponential Functions

F. Theory of Numbers

G. Higher Algebra

H. Numerical Solution of Equations

I. Finite Differences. Interpolation

J. Summation of Series

K. Statistics

L. Higher Mathematical Functions

M. Integrals

N. Interest and Investment

O. Actuarial Science

P. Engineering

Q. Astronomy

R. Geodesy

S. Physics, Geophysics, Crystallography

T. Chemistry

U. Navigation

V. Aerodynamics, Hydrodynamics, Ballistics

Z. Calculating Machines and Mechanical Computation

CONTENTS

APRIL 1953

A Technique for Real Time Simulation of a Rigid Body Problem.... .....H. J. GRAY, JR., M. RUBINOFF & H. SOHON 73

A Solution of Simultaneous Linear Equations and Matrix Inversion with High Speed Computing Devices..... .....G. G. BEJARANO & B. R. ROSENBLATT 77

Error Bounds on Approximate Solutions to Systems of Linear Algebraic Equations.....A. DE LA GARZA 81

Recent Mathematical Tables..... 84

AERONAUTICAL RESEARCH COUNCIL **1093,** ANDERSON & DARLING **1071,** BIRNBAUM **1072,** BOWKER & GOODE **1073,** BAAS **1087,** FREUDENBERG **1074,** GOULDEN **1075,** GUPTA **1067,** HALD **1076,** HAMMERSLEY **1070,** HEMER **1068,** KITAGAWA **1077,** KOBER **1088,** KRAITCHIK **1069,** KUNISAWA, MAKABE & MORIMURA **1078,** MARCH **1089,** MASSEY **1079,** MAY **1080,** MEYER ZUR CAPELLEN **1090,** MOSHMAN **1081,** NANDA **1082,** NBS **1091,** PILLAI **1083,** RSMTC **1092,** STUART **1084,** TERPSTRA **1085,** UHLER **1065,** WISHART **1086,** WITTKE **1066.**

Mathematical Tables—Errata..... 104

LJUSTERNIK, AKUSHSKIĬ & DITKIN **221,** MCLACHLAN, HUMBERT & POLI **222,** MEYER ZUR CAPELLEN **223,** NBSMTP **224,** NICHOLSON & AIREY **225,** SPENCELEY & SPENCELEY **226.**

Unpublished Mathematical Tables..... 108

GLODEN **158,** PAGE & PFEIL **156,** WRENCH **157.**

Automatic Computing Machinery..... 108

Technical Developments, The Electronic Computer at the Institute for Advanced Study.....GERALD ESTRIN 108

Bibliography Z: DE FINETTI **1016,** GRUENBERGER **1017,** HUSKEY & HUSKEY **1018,** KENOSIAN **1019,** KOPPEL **1020,** MULLANEY **1021,** ONR **1022,** PACKER & RAY **1023,** RABINOW **1024.** 115

News..... 118

Other Aids to Computation..... 119

Bibliography Z: ALTMAN **1025,** BOSWORTH **1026,** BROOKS & SMITH **1027,** DANIELS **1028,** GOLDBERG **1029,** HASTINGS & MEADE **1030,** INST. OF GAS TECH. **1031,** MINDLIN & SALVADORI **1032,** REEVES INST. CO. **1033,** SHAW & WELTMANN **1034,** SOFTKY & JUNGERMAN **1035,** VANCE & SHUMARD **1036,** WEISSLER, EINARSSON & MCCLELLAND **1037.**

Notes..... 127

147. Stability of Difference Relations in the Solution of Ordinary Differential Equations....A. R. MITCHELL & J. W. CRAGGS 127

148. Two Non-Elementary Definite Integrals.M. S. CORRINGTON 129

149. The Canon Doctrinae Triangvlorvm (1551) of Rheticus (1514–1576).....R. C. ARCHIBALD 131

150. Square Root on the 602A..... .....J. HOLLINGSWORTH & F. GRUENBERGER 132

151. A Numerical Study of a Conjecture by Kummer..... .....J. VON NEUMANN & H. H. GOLDSTINE 133

Corrigenda..... 134

LANCASTER PRESS, INC., LANCASTER, PA.

.. 73
on
.. 77
ic
ZA 81
.. 84
NG
U-
M-
T-
39,
AN
RT
56.
.. 104
RT
ON
.. 108
.. 108
ti-
IN 108
& 115
21,
.. 118
.. 119
&
DE
32,
&
ON
.. 127
ry
GS 127
ON 129
us
LD 131
..
ER 132
..
NE 133
.. 134